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Maximum Likelihood Estimation for Step-Stress Partially Accelerated Life Tests based on Censored Data

N CHANDRA* AND MASHROOR AHMAD KHAN†

Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, Pondicherry-605 014, India

*Email: nc.stat@gmail.com; †Email: mashroor08@gmail.com

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Abstract: The aim of this article is to perform the estimation procedures on Rayleigh parameter in step-stress partially accelerated life tests (PALT) under both Type-I and Type-II censored samples in which all the test units are first run simultaneously under normal conditions for a pre-specified time, and the surviving units are then run under accelerated conditions until a predetermined censoring reached. It is assumed that the lifetime of the test units follows Rayleigh distribution. The maximum likelihood estimates are obtained for the proposed model parameters and acceleration factor for each of Type-I and Type-II censored data. In addition, the asymptotic variances and covariance matrix of the estimators are presented, and confidence intervals of the estimators are also given.

Keywords: Step-stress partially accelerated life tests, Rayleigh distribution, Maximum likelihood estimation, Confidence interval, Type-I and Type-II censoring.

1. Introduction

The concept of accelerated life testing (ALT) was first introduced by [12] and Bessler [6]. The main purpose of using ALT is to collect the sufficient failure time data of life testing units in shorter period of time. Because many manufactured units/ products have a long life and standard life testing of such units are time consuming, very expensive and may not be purposeful. Therefore, ALT is recommended to use. [17] first elaborated the concept of step-stress ALT, in which the stress can be applied in different ways, commonly used methods are constant-stress, step-stress and progressive-stress. Many

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other authors like; [18], [8], [16], [15], [9], [10] and [11] gave some more applications and attempted the work in this direction.

In the case of ALT, the acceleration factor is assumed as a known value or there is a known mathematical model which specifies the relationship between lifetime and stress conditions. But in some situations such life-stress relationship are not known and cannot be assumed. Therefore, in such cases, partially accelerated life tests (PALTs) are better criterion to perform life test to estimate the acceleration factor and parameters of the life distribution. The concept of PALT was firstly introduced by [14] in which a test unit is first run at use condition and if it does not fail for a pre-specified time ‘τ’, the test is switched to the higher level of stress for testing until all the unit fails or censoring reached. The effect of this switch is to multiply the remaining lifetime of the unit by an unknown factor which is called acceleration factor β. Thus, the total lifetime T of test unit is given by

$$T = \begin{cases} Y, & Y \le \tau \\ \tau + \beta^{-1}\left(Y - \tau\right), & Y > \tau \end{cases} \tag{1.1}$$

where Y denotes the lifetime of unit at normal use condition.

[14] considered the estimation problem using maximum likelihood (ML) and Bayesian methods for estimating the parameters of the Exponential and Uniform distribution. [13] studied the problem of estimation for acceleration factor and Exponential parameters by using Bayesian approach with different loss functions for complete data set in step-stress PALT. [7] also estimated the parameters of the Weibull distribution and acceleration factor using ML method in step-stress PALT. [4] reported ML method for estimating the acceleration factor and scale parameter of Exponential distribution under type-1 censoring. [5] estimated parameters of the lognormal distribution and acceleration factor using ML method under type-1 censored data. Recently, [20] developed ML method for estimating the parameters and acceleration factor of the Weibull distribution under multiply censored data. For more details, see [3], [2] and [1].

This paper deals with the step-stress PALT for Rayleigh distribution under Type-I and Type-II censored case. Where the performance of the parameter estimators are investigated on the basis of mean square error (MSE), relative absolute bias (RABias) and relative error (RE) based on simulation data. Moreover, the asymptotic variances and covariance matrix and confidence interval of the estimators are obtained.

In addition to this introductory section this article includes some more sections too. In section 2 the proposed model and assumption are described. Section 3 presents maximum likelihood estimation (MLEs) under Type-I and

Type-II censoring. In section 4 confidence intervals of the model parameter and acceleration factor are described. Simulation studies to illustrate the theoretical results are given in section 5. Finally, the conclusion of the study is discussed in section 6.

2. The Model Description and Assumptions

This section describes the notation and introduces the assumed model for product life and test procedure also.

Notations used

n	: total number of test an items in a step-stress PALT
T	: lifetime of an item at use conditions
Y	: total lifetime of an item in a step-stress PALT
y_i	: observed value of the total lifetime Y_i of item i, i=1, 2, ……n
Yc	: censoring time of a PALT in the case of Type-I censoring
n_c	: number of censored items in the case of Type-I censoring
r	: number of failures at which the test is terminated in case of Type-II censoring
$y_{(r)}$	: the time of the r^{th} failure at which the test is terminated in case of Type-II censoring
β	: acceleration factor (β>1)
θ	: scale parameter (θ>0)
τ	: stress changing time in step PALT, that is, τ < Yc in case of Type-I censoring and τ < $y_{(r)}$ in case of Type-II censoring
n_u, n_a	: number of items failed at normal and accelerated condition respectively

$y_{(1)} \leq ..... \leq y_{(n_u)} \leq \tau \leq y_{(n_u+1)} \leq ..... \leq y_{(n_u+n_a)} \leq Y_c$ and $y_{(1)} \leq ..... \leq y_{(n_u)} \leq \tau \leq y_{(n_u+1)} \leq ..... \leq y_{(r)}$ are the ordered failure times in case of Type-I and Type-II censoring, respectively.

The Rayleigh distribution has played an important role in the modeling the lifetime of random process and having many applications, including reliability, life testing and survival analysis. The probability density function (p.d.f.) is given below as

$$f(t;\theta) = \frac{t}{\theta^2} \exp\left(-\frac{t^2}{2\theta^2}\right), \quad t > 0, \theta > 0 \tag{2.1}$$

and the cumulative density function is

$$F(t,\theta)=1-\exp\left(-\frac{t^2}{2\theta^2}\right), \quad t>0, \theta>0 \qquad (2.2)$$

The Reliability function is

$$R(t)=\exp\left(-\frac{t^2}{2\theta^2}\right), \qquad t>0,\ \theta>0 \qquad (2.3)$$

Assumptions

(a) The total lifetime Y of an item is defined as

$$T=\begin{cases} Y, & Y\le\tau \\ \tau+\beta^{-1}(Y-\tau), & Y>\tau \end{cases} \qquad (2.4)$$

(b) The lifetime of an item tested at both use and at accelerated condition follows Rayleigh distribution.
(c) The lifetimes of test items are independent and identically distributed random variables.
(d) Under Type-I censoring, the test terminates when the censoring time 'Yc' is reached.
(e) Under Type-II censoring, the test terminates when the predetermined number of failures 'r' is reached.

3. Maximum Likelihood Estimation

In this section the MLEs of the acceleration factor and scale parameter in step-stress PALT are obtained under Type-I and Type-II censoring.

The lifetime of test unit is assumed to fellow the Rayleigh distribution with p.d.f. given in equation (1). Therefore, the p.d.f. of the total lifetime Y of an item in step-stress PALT is given by

$$f(y)=\begin{cases} f_1(y), & \text{if } 0<y\le\tau \\ f_2(y), & \text{if } y>\tau \end{cases} \qquad (3.1)$$

Where

$f_1(y)=\frac{y}{\theta^2}\exp\left(-\frac{y^2}{2\theta^2}\right),\ \theta>0$, is equivalent form to equation (2.1), and

$$f_2(y)=\beta\frac{[\tau+\beta(y-\tau)]}{\theta^2}\exp\left(-\frac{[\tau+\beta(y-\tau)]^2}{2\theta^2}\right), \quad \theta>0, \beta>1$$

3.1 The case of type-I censoring

The observed values of the total lifetime Y are given by

$$y_{(1)} \le ..... \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le ..... \le y_{(n_u+n_a)} \le Y_c$$

Since the total lifetimes Y_1……,Y_n of n items are i.i.d. random variables, then the likelihood function for them can be written as

$$L(y;\beta,\theta) = \prod_{i=1}^{n_u} f_1(y_i) \prod_{i=1}^{n_a} f_2(y_i) \prod_{i=1}^{n_c} R(Y_c) \tag{3.2}$$

where, $n = n_u + n_a + n_c$

Therefore, after substituting the values of f_1(y), f_2(y) and R(Y_c), the likelihood function of the sample is given by

$$L(y;\beta,\theta) = \prod_{i=1}^{n_u} \frac{y_i}{\theta^2} \exp\left(-\frac{y_i^2}{2\theta^2}\right) \prod_{i=1}^{n_a} \beta \frac{\left[\tau + \beta(y_i - \tau)\right]}{\theta^2} \times \exp\left(-\frac{\left[\tau + \beta(y_i - \tau)\right]^2}{2\theta_2}\right) \prod_{i=1}^{n_c} \exp\left(-\frac{\left[\tau + \beta(Y_c - \tau)\right]^2}{2\theta_2}\right) \tag{3.3}$$

On maximizing the natural logarithm of the equation (3.3), the maximum likelihood estimates of β and θ can be obtained. After taking the log of above equation (3.3), it can be written as

$$\log L(\beta,\theta;y) = \sum_{i=1}^{n_u} \log\left(\frac{y_i}{\theta^2} \exp\left(-\frac{y_i^2}{2\theta^2}\right)\right) + n_c \exp\left(-\frac{\left[\tau + \beta(Y_c - \tau)\right]^2}{2\theta_2}\right) + \sum_{i=1}^{n_a} \log\left(\beta \frac{\left[\tau + \beta(y_i - \tau)\right]}{\theta^2} \exp\left(-\frac{\left[\tau + \beta(y_i - \tau)\right]^2}{2\theta^2}\right)\right) \tag{3.4}$$

$$\log L(\beta,\theta;y) = -2n_0 \log\theta + n_a \log\beta + \sum_{i=1}^{n_u} \log y_i - \frac{1}{2\theta^2} \sum_{i=1}^{n_u} y_i^2 + \sum_{i=1}^{n_a} \log\left[\tau + \beta(y_i - \tau)\right] - \frac{1}{2\theta^2} \sum_{i=1}^{n_a} \left[\tau + \beta(y_i - \tau)\right]^2 - \frac{n_c}{2\theta^2}\left[\tau + \beta(Y_c - \tau)\right]^2 \tag{3.5}$$

The first order partial derivatives of Eq. (3.5) with respect to β and θ are given by

$$\frac{\partial \log L(\beta,\theta)}{\partial\beta} = \frac{n_a}{\beta} + \sum_{i=1}^{n_a} \frac{(y_i - \tau)}{[\tau + \beta(y_i - \tau)]} - \frac{1}{\theta^2}\sum_{i=1}^{n_a}(y_i - \tau)[\tau + \beta(y_i - \tau)] - \frac{n_c}{\theta^2}(Y_c - \tau)[\tau + \beta(Y_c - \tau)] = 0 \tag{3.6}$$

$$\frac{\partial \log L(\beta,\theta)}{\partial\theta} = -\frac{n_0}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n_u} y_i^2 + \frac{1}{\theta^3}\sum_{i=1}^{n_a}[\tau + \beta(y_i - \tau)]^2 + \frac{n_c}{\theta^3}[\tau + \beta(Y_c - \tau)]^2 = 0 \tag{3.7}$$

where, $n_0 = n_u + n_a$

We observe that likelihood equations (3.6) and (3.7) are difficult to solve as these are functions of population parameters which are themselves functions of the solutions of these equations. Due to this difficulty, it is not possible to find exact solutions. We shall therefore, find MLE solutions $(\hat{\beta},\hat{\theta})$ through iterative procedure.

The asymptotic variances and covariance of the estimates are given by the elements of the inverse of the Fisher information matrix. Since, the exact mathematical expression for the expectation is too difficult to find. So, it can be approximated by numerically inverting the asymptotic Fisher information matrix, which is obtained from the negative of second and mixed partial derivatives of the natural logarithm of the likelihood function evaluated at the estimates of the parameters. So, asymptotic Fisher information matrix can be written as

$$F = \begin{bmatrix} -\frac{\partial^2 \log L}{\partial\beta^2} & -\frac{\partial^2 \log L}{\partial\beta\partial\theta} \\ \frac{\partial^2 \log L}{\partial\beta\partial\theta} & \frac{\partial^2 \log L}{\partial\theta^2} \end{bmatrix}_{(\beta=\hat{\beta},\,\theta=\hat{\theta})} \tag{3.8}$$

The elements of the above matrix F can be expressed by the following equations

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \beta^2} = -\frac{n_a}{\beta^2} - \sum_{i=1}^{n_a} \frac{(y_i-\tau)^2}{\left[\tau+\beta(y_i-\tau)\right]^2} - \frac{1}{\theta^2}\sum_{i=1}^{n_a}(y_i-\tau)^2 - \frac{n_c}{\theta^2}(Y_c-\tau)^2 \tag{3.9}$$

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \theta^2} = -\frac{n_0}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n_u} y_i^2 - \frac{3}{\theta^4}\sum_{i=1}^{n_a}\left[\tau+\beta(y_i-\tau)\right]^2 - n_c\frac{3}{\theta^4}\left[\tau+\beta(Y_c-\tau)\right]^2 \tag{3.10}$$

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \beta \partial \theta} = \frac{2}{\theta^3}\sum_{i=1}^{n_a}(y_i-\tau)\left[\tau+\beta(y_i-\tau)\right] + n_c\frac{2}{\theta^3}(Y_c-\tau)\left[\tau+\beta(Y_c-\tau)\right] \tag{3.11}$$

3.2 The case of type-II censoring

The observed values of the total lifetime Y are given by

$y_{(1)} \le ..... \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le ..... \le y_{(r)}$

Since the total lifetimes $Y_1,\ldots\ldots,Y_n$ of n items are i.i.d. random variables, then the likelihood function for them can be written as

$$L(y;\beta,\theta) = \prod_{i=1}^{n_u} f_1(y_i)\prod_{i=1}^{n_a} f_2(y_i)\prod_{i=1}^{n-r} R\left(y_{(r)}\right) \tag{3.12}$$

where, $r = n_u + n_a$

Therefore, after substituting the values of $f_1(y)$, $f_2(y)$ and $R(y_{(r)})$, the likelihood function of the sample is given by

$$L(y;\beta,\theta) = \prod_{i=1}^{n_u}\frac{y_i}{\theta^2}\exp\left(-\frac{y_i^2}{2\theta^2}\right)\prod_{i=1}^{n_a}\beta\frac{\left[\tau+\beta(y_i-\tau)\right]}{\theta^2} \times\exp\left(-\frac{\left[\tau+\beta(y_i-\tau)\right]^2}{2\theta_2}\right) \prod_{i=1}^{n-r}\exp\left(-\frac{\left[\tau+\beta\left(y_{(r)}-\tau\right)\right]^2}{2\theta_2}\right) \tag{3.13}$$

On maximizing the natural logarithm of the above equation (3.13), the maximum likelihood estimates of β and θ can be obtained. After taking the log of above equation (3.13), it can be written as

$$\begin{aligned}\log L(\beta,\theta;y) = &\sum_{i=1}^{n_u} \log\left(\frac{y_i}{\theta^2}\exp\left(-\frac{y_i^2}{2\theta^2}\right)\right) \\ &+(n-r)\exp\left(-\frac{\left[\tau+\beta\left(y_{(r)}-\tau\right)\right]^2}{2\theta_2}\right) \\ &+\sum_{i=1}^{n_a}\log\left(\beta\frac{\left[\tau+\beta\left(y_i-\tau\right)\right]}{\theta^2}\exp\left(-\frac{\left[\tau+\beta\left(y_i-\tau\right)\right]^2}{2\theta^2}\right)\right)\end{aligned} \tag{3.14}$$

$$\begin{aligned}\log L(\beta,\theta;y) = &-2n_0\log\theta + n_a\log\beta + \sum_{i=1}^{n_u}\log y_i \\ &-\frac{1}{2\theta^2}\sum_{i=1}^{n_u} y_i^2 + \sum_{i=1}^{n_a}\log\left[\tau+\beta\left(y_i-\tau\right)\right] \\ &-\frac{1}{2\theta^2}\sum_{i=1}^{n_a}\left[\tau+\beta\left(y_i-\tau\right)\right]^2 - \frac{n-r}{2\theta^2}\left[\tau+\beta\left(y_{(r)}-\tau\right)\right]^2\end{aligned} \tag{3.15}$$

The first order partial derivatives of Eq. (3.20) with respect to β and θ are given by

$$\begin{aligned}\frac{\partial\log L(\beta,\theta)}{\partial\beta} = &\frac{n_a}{\beta} + \sum_{i=1}^{n_a}\frac{\left(y_i-\tau\right)}{\left[\tau+\beta\left(y_i-\tau\right)\right]} \\ &-\frac{1}{\theta^2}\sum_{i=1}^{n_a}\left(y_i-\tau\right)\left[\tau+\beta\left(y_i-\tau\right)\right] \\ &-\frac{n-r}{\theta^2}\left(y_{(r)}-\tau\right)\left[\tau+\beta\left(y_{(r)}-\tau\right)\right]=0\end{aligned} \tag{3.16}$$

$$\begin{aligned}\frac{\partial\log L(\beta,\theta)}{\partial\theta} = &-\frac{n_0}{\theta} + \frac{1}{\theta^3}\sum_{i=1}^{n_u} y_i^2 + \frac{1}{\theta^3}\sum_{i=1}^{n_a}\left[\tau+\beta\left(y_i-\tau\right)\right]^2 \\ &+\frac{n-r}{\theta^3}\left[\tau+\beta\left(y_{(r)}-\tau\right)\right]^2 = 0\end{aligned} \tag{3.17}$$

where, $n_0 = n_u + n_a$

Obviously, it is very difficult to obtain a closed form solution for two non-linear equations (3.21) and (3.22). We shall therefore, find MLE solutions $\left(\hat{\beta},\hat{\theta}\right)$ through iterative procedure.

Proceeding the similar way as in case of Type-I censoring, the elements of the observed Fisher information matrix are described by the following equations

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \beta^2} = -\frac{n_a}{\beta^2} - \sum_{i=1}^{n_a} \frac{(y_i - \tau)^2}{\left[\tau + \beta(y_i - \tau)\right]^2} - \frac{1}{\theta^2}\sum_{i=1}^{n_a}(y_i - \tau)^2 - \frac{n-r}{\theta^2}\left(y_{(r)} - \tau\right)^2 \tag{3.18}$$

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \theta^2} = -\frac{n_0}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n_u} y_i^2 - \frac{3}{\theta^4}\sum_{i=1}^{n_a}\left[\tau + \beta(y_i - \tau)\right]^2 - (n-r)\frac{3}{\theta^4}\left[\tau + \beta\left(y_{(r)} - \tau\right)\right]^2 \tag{3.19}$$

$$\frac{\partial^2 \log L(\beta,\theta)}{\partial \beta \partial \theta} = \frac{2}{\theta^3}\sum_{i=1}^{n_a}(y_i - \tau)\left[\tau + \beta(y_i - \tau)\right] + (n-r)\frac{2}{\theta^3}\left(y_{(r)} - \tau\right)\left[\tau + \beta\left(y_{(r)} - \tau\right)\right] \tag{3.20}$$

4. Confidence Intervals for the Case of Type-I and Type-II Censoring

[19] indicate that the most common method to construct confidence bounds for the parameter is to use the large sample normal distribution of the maximum likelihood estimators.

To construct a confidence interval for a population parameter α; assume that $L_\alpha = L_\alpha(y_1......y_n)$ and $U_\alpha = U_\alpha(y_1.......y_n)$ are the functions of the sample data $y_1, y_2, ..., y_n$ such that

$$P_\alpha = (L_\alpha \le y \le U_\alpha) = \varepsilon \tag{4.1}$$

Where, L_α and U_α are indicating the lower and upper confidence limits which enclose α with probability ε. The interval $[L_\alpha, U_\alpha]$ is called a two sided 100 ε % confidence interval for α.

It is known that the MLEs, for large sample size under appropriate regularity conditions, are consistent and normally distributed. Therefore, the two-sided approximate 100 ε% confidence limits for a population parameter can be constructed as follows:

$$P\left[-z \le \frac{\hat{\alpha}-\alpha}{\sigma(\hat{\alpha})} \le z\right] \cong \varepsilon \tag{4.2}$$

where, z is the $[100(1-\varepsilon)/2]^{th}$ percentile of the standard normal. Therefore, the two-sided approximate ε100% confidence limits for β and θ are given respectively as follows:

$$\left.\begin{array}{ll} L_\beta = \hat{\beta} - z\sqrt{Var(\hat{\beta})}, & U_\beta = \hat{\beta} + z\sqrt{Var(\hat{\beta})} \\ L_\theta = \hat{\theta} - z\sqrt{Var(\hat{\theta})}, & U_\theta = \hat{\theta} + z\sqrt{Var(\hat{\theta})} \end{array}\right\} \tag{4.3}$$

5. Simulation Studies

The simulation studies have been performed using R software for illustrating the theoretical results of estimation problem. The performance of the resulting estimators of the acceleration factor and scale parameters has been considered in terms of their MSE, RABias and RE. Furthermore, the asymptotic variances and covariance matrix and confidence intervals of the acceleration factor and scale parameter are obtained. The simulation procedures were performed in following steps as

Step 1: 1000 random samples of sizes 50(50) 400 and 500 were generated from Rayleigh distribution. The generation of the Rayleigh distribution is very simple, if U has a uniform (0, 1) random number, and then $Y = \left[-2\theta^2 \log(1-U)^{1/2}\right]$ follows a Rayleigh distribution. The true parameter values are chooses as (β=1.25, θ=2) and (β=1.75, θ=1.5) in case of Type-I censoring and (β=1.25, θ=2) and (β=1.75, θ=1.8) for Type-II censoring.

Step 2: Choosing the stress changing time τ at normal condition to be τ=2 and censoring time Y_c=5 in case of Type-I censoring and the total number of failure in the test of a PALT to be r=0.75n in case of Type-II censoring.

Step 3: For each sample and for the two sets of parameters, the acceleration factor and the scale parameters of distribution were estimated in PALT under Type-I and Type-II censored sample by using *optim()* function in R software.

Step 4: The RABias, MSE, and RE of the estimators for acceleration factor and scale parameter for all sample sizes and for two sets of parameters were tabulated.

Step 5: The asymptotic variance and covariance matrix of the estimators for different sample sizes were obtained.

Step 6: The confidence limit with confidence level = 0.95 γ and = 0.99 γ of the acceleration factor and scale parameters were constructed.

Simulation results are summarized in Tables 1-6. Tables 1-3 represents the findings for the case Type-I censoring, in which Table 1 gives the MSE, RABias and RE of the estimators, the asymptotic variances and covariance matrix of the estimators are given in Table 2 and the approximated confidence limits at 95% and 99% confidence level are presented in Table 3.

Similarly, Tables 4-6 represents the results for the case of Type-II censoring, where MSE, RABias and RE are presented in Table 4, the asymptotic variances and covariance matrix of the estimators are given in Table 5 and the approximated confidence limits at 95% and 99% confidence level are presented in Table 6.

Table 1: The MSE, RABias and RE of the parameters (β, θ, τ) given Y_c=5 for different samples sizes under Type-I censoring

Sample size	Parameters (β, θ, τ)	True value of parameters (1.25, 2, 2)			True value of parameters (1.75, 1.5, 2)		
		MSE	RABias	RE	MSE	RABias	RE
50	β	0.08979	0.05187	0.23973	0.21987	0.06043	0.26795
	θ	0.05858	0.01920	0.12101	0.02155	0.00806	0.09786
100	β	0.03933	0.01876	0.15866	0.09142	0.02618	0.17278
	θ	0.02749	0.00551	0.08290	0.01019	0.00183	0.06730
150	β	0.02725	0.01828	0.13206	0.05467	0.02248	0.13361
	θ	0.01799	0.00539	0.06707	0.00671	0.00483	0.05461
200	β	0.01955	0.01220	0.11187	0.03974	0.01512	0.11391
	θ	0.01319	0.00274	0.05743	0.00469	0.00365	0.04567
250	β	0.01535	0.01308	0.09911	0.02947	0.00992	0.09810
	θ	0.01062	0.00555	0.05153	0.00379	0.00435	0.04105
300	β	0.01195	0.00803	0.08746	0.02545	0.00607	0.09116
	θ	0.00856	0.00502	0.04626	0.00304	0.00003	0.03677
350	β	0.01012	0.00395	0.08049	0.02369	0.00635	0.08795
	θ	0.00708	0.00122	0.04206	0.00281	0.00080	0.03531
400	β	0.01011	0.00601	0.08043	0.02169	0.00488	0.08415
	θ	0.00665	0.00209	0.04078	0.00250	0.00118	0.03336
450	β	0.00865	0.00431	0.07441	0.01674	0.00245	0.07392
	θ	0.00550	0.00069	0.03709	0.00223	0.00087	0.03151
500	β	0.00714	0.00453	0.06762	0.01525	0.00521	0.07057
	θ	0.00538	0.00092	0.03666	0.00188	0.00067	0.02890

Table 2: Asymptotic variances and covariance of the estimates under Type-I censoring

Sample size	(1.25, 2, 2)		(1.75, 1.5, 2)	
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$
50	0.03370	0.00673	0.01163	0.00074
	----	0.00358	----	0.00280
100	0.01666	0.00340	0.00574	0.00036
	----	0.00181	----	0.00132
150	0.01111	0.00227	0.00386	0.00025
	----	0.00121	----	0.00088
200	0.00831	0.00170	0.00289	0.00018
	----	0.00091	----	0.00066
250	0.00671	0.00137	0.00231	0.00015
	----	0.00072	----	0.00052
300	0.00562	0.00115	0.00191	0.00012
	----	0.00060	----	0.00043
350	0.00477	0.00098	0.00164	0.00010
	----	0.00052	----	0.00037
400	0.00418	0.00086	0.00143	0.00009
	----	0.00045	----	0.00032
450	0.00368	0.00076	0.00128	0.00008
	----	0.00040	----	0.00029
500	0.00333	0.00069	0.00115	0.00007
	----	0.00036	----	0.00026

Table 3: Confidence bounds of the estimates at 0.95 and 0.99 confidence level under Type-I censoring

Sample size	Parameter	(1.25, 2, 2)			(1.75, 1.5, 2)		
		SD	L	U	SD	L	U
50	β	0.18358	0.95502 0.84121	1.67464 1.78846	0.10784	1.64438 1.57752	2.06713 2.13399
	θ	0.05983	1.92113 1.88403	2.15568 2.19277	0.05292	1.40837 1.37556	1.61580 1.64861
100	β	0.12907	1.02046 0.94044	1.52643 1.60646	0.07576	1.64731 1.60034	1.94430 1.99128
	θ	0.04254	1.92763 1.90126	2.09441 2.12078	0.03633	1.43153 1.40900	1.57395 1.59648

150	β	0.10540	1.06626 1.00091	1.47945 1.54480	0.06213	1.66757 1.62905	1.91111 1.94963
	θ	0.03479	1.94260 1.92104	2.07896 2.10053	0.02966	1.44910 1.43071	1.56539 1.58378
200	β	0.09116	1.08658 1.03006	1.44392 1.50044	0.05376	1.67109 1.63776	1.88183 1.91516
	θ	0.03017	1.94635 1.92765	2.06460 2.08331	0.02569	1.45512 1.43919	1.55582 1.57175
250	β	0.08191	1.10579 1.05501	1.42690 1.47769	0.04806	1.67316 1.64337	1.86157 1.89137
	θ	0.02683	1.95852 1.94188	2.06370 2.08034	0.02280	1.46182 1.44769	1.55121 1.56535
300	β	0.07497	1.11310 1.06662	1.40697 1.45345	0.04370	1.67496 1.64786	1.84628 1.87337
	θ	0.02449	1.96203 1.94684	2.05805 2.07324	0.02074	1.45940 1.44655	1.54069 1.55355
350	β	0.06907	1.11957 1.07675	1.39031 1.43313	0.04050	1.68174 1.65663	1.84049 1.86559
	θ	0.02280	1.95774 1.94360	2.04713 2.06127	0.01924	1.46350 1.45157	1.53890 1.55083
400	β	0.06465	1.13079 1.09071	1.38423 1.42432	0.03782	1.68443 1.66098	1.83266 1.85611
	θ	0.02121	1.96260 1.94945	2.04576 2.05891	0.01789	1.46318 1.45208	1.53330 1.54439
450	β	0.06066	1.13649 1.09888	1.37429 1.41190	0.03578	1.68417 1.66199	1.82442 1.84660
	θ	0.02000	1.95941 1.94701	2.03781 2.05021	0.01703	1.46793 1.45738	1.53469 1.54525
500	β	0.05771	1.14256 1.10679	1.36877 1.40455	0.03391	1.69264 1.67161	1.82557 1.84660
	θ	0.01897	1.96465 1.95288	2.03903 2.05079	0.01612	1.46838 1.45838	1.53159 1.54158

The first entire of each parameter is for 95% significance level and second for 99%.

Table 4: The MSE, RABias and RE of the parameters (β, θ τ) given r=0.75*n for different samples sizes under Type-II censoring.

Sample size	**Parameters (β, θ, τ)**	**True value of parameters (1.25, 2, 2)**			**True value of parameters (1.75, 1.8, 2)**		
		MSE	**RABias**	**RE**	**MSE**	**RABias**	**RE**
50	β	0.15100	0.08165	0.31087	0.31520	0.08519	0.32082
	θ	0.05555	0.00569	0.11785	0.03753	0.00476	0.10762

100	β	0.06200	0.04246	0.19920	0.15396	0.04641	0.22422
	θ	0.02691	0.00958	0.08202	0.01852	0.00670	0.07561
150	β	0.04021	0.02643	0.16043	0.09097	0.02468	0.17235
	θ	0.01780	0.00038	0.06671	0.01272	0.00049	0.06265
200	β	0.02787	0.01524	0.13356	0.07136	0.02474	0.15265
	θ	0.01326	0.00158	0.05757	0.00913	0.00408	0.05308
250	β	0.02556	0.01606	0.12791	0.05319	0.01772	0.13179
	θ	0.00975	0.00012	0.04937	0.00723	0.00007	0.04723
300	β	0.02043	0.01689	0.11435	0.04065	0.01377	0.11521
	θ	0.00893	0.00525	0.04725	0.00609	0.00147	0.04336
350	β	0.01616	0.01114	0.10169	0.03580	0.01156	0.10812
	θ	0.00744	0.00009	0.04312	0.00501	0.00029	0.03932
400	β	0.01347	0.00709	0.09285	0.03105	0.01554	0.10069
	θ	0.00613	0.00144	0.03913	0.00409	0.00180	0.03554
450	β	0.01171	0.00692	0.08658	0.02780	0.01162	0.09528
	θ	0.00537	0.00013	0.03663	0.00421	0.00200	0.03607
500	β	0.01095	0.00691	0.08371	0.02258	0.00905	0.08586
	θ	0.00498	0.00017	0.03528	0.00348	0.00369	0.03275

Table 5: Asymptotic variances and covariance of the estimates under Type-II censoring

Sample size	(1.25, 2, 2)		(1.75, 1.5, 2)	
	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$
50	0.02963	0.00534	0.02163	0.00649
	----	0.00679	----	0.01999
100	0.01534	0.00279	0.01169	0.00315
	----	0.00275	----	0.01039
150	0.01048	0.00163	0.00747	0.00211
	----	0.00230	----	0.00618
200	0.00769	0.00129	0.00591	0.00156
	----	0.00151	----	0.00534
250	0.00612	0.00104	0.00455	0.00126
	----	0.00121	----	0.00387
300	0.00527	0.00082	0.00399	0.00104
	----	0.00107	----	0.00378
350	0.00443	0.00072	0.00335	0.00089
	----	0.00090	----	0.00304
400	0.00384	0.00066	0.00285	0.00078
	----	0.00073	----	0.00236
450	0.00345	0.00056	0.00260	0.00069
	----	0.00070	----	0.00229
500	0.00308	0.00051	0.00235	0.00062
	----	0.00060	----	0.00203

Table 6: Confidence bounds of the estimates at 0.95 and 0.99 confidence level under Type-II censoring

Sample size	Parameter	(1.25, 2, 2)			(1.75, 1.5, 2)		
		SD	**L**	**U**	**SD**	**L**	**U**
50	β	0.17213	1.01468 0.90796	1.68945 1.79617	0.14707	1.61082 1.51963	2.18734 2.27852
	θ	0.08240	1.82711 1.77603	2.15013 2.20122	0.14139	1.51431 1.42665	2.06854 2.15620
100	β	0.12385	1.06031 0.98352	1.54582 1.62261	0.10812	1.61931 1.55227	2.04314 2.11017
	θ	0.05244	1.91637 1.88386	2.12194 2.15445	0.10193	1.61228 1.54908	2.01185 2.07505
150	β	0.10237	1.08239 1.01892	1.48369 1.54716	0.08643	1.62379 1.57020	1.96259 2.01618
	θ	0.04796	1.90675 1.87702	2.09475 2.12448	0.07861	1.64680 1.59806	1.95497 2.00371
200	β	0.08769	1.09717 1.04280	1.44093 1.49530	0.07688	1.64261 1.59495	1.94397 1.99163
	θ	0.03886	1.92700 1.90291	2.07933 2.10342	0.07308	1.66411 1.61880	1.95057 1.99587
250	β	0.07823	1.11675 1.06824	1.42341 1.47191	0.06745	1.64880 1.60698	1.91322 1.95504
	θ	0.03479	1.93206 1.91049	2.06841 2.08998	0.06221	1.67820 1.63963	1.92206 1.96063
300	β	0.07259	1.12882 1.08381	1.41339 1.45840	0.06317	1.65029 1.61113	1.89791 1.93707
	θ	0.03271	1.94638 1.92610	2.07461 2.09489	0.06148	1.68214 1.64402	1.92314 1.96126
350	β	0.06656	1.13347 1.09220	1.39438 1.43565	0.05788	1.65679 1.62091	1.88368 1.91956
	θ	0.03000	1.94137 1.92277	2.05897 2.07757	0.05514	1.69245 1.65826	1.90858 1.94277
400	β	0.06197	1.13741 1.09899	1.38032 1.41874	0.05339	1.67256 1.63946	1.88183 1.91493
	θ	0.02702	1.94992 1.93316	2.05583 2.07258	0.04858	1.70802 1.67790	1.89845 1.92857
450	β	0.05874	1.14353 1.10711	1.37378 1.41019	0.05099	1.67039 1.63877	1.87027 1.90188
	θ	0.02646	1.94840 1.93200	2.05212 2.06852	0.04785	1.70980 1.68013	1.89739 1.92706
500	β	0.05550	1.14987 1.11546	1.36742 1.40183	0.04848	1.67082 1.64076	1.86085 1.89090
	θ	0.02449	1.95234 0.93715	2.04836 2.06355	0.04506	1.71833 1.69040	1.89495 1.92289

The first entire of each parameter is for 95% significance level and second for 99%.

5.1 Findings

From these tables, we can be made the following observations on the performance of SS-PALT parameter estimation of the Rayleigh distribution:

1. For the second set of parameters (β=1.75, θ=1.5), the maximum likelihood estimators have good statistical properties than the first set of parameters (β =1.25, θ =2) for all sample size for Type-I censoring (Table 1); and for Type-II censoring the first set of parameters (β =1.25, θ =2), the maximum likelihood estimators have good statistical properties than the second set of parameters (β = 1.75, θ=1.8) for all sample size (Table 4).
2. As the sample sizes increases the MSE and RABias of the estimated parameters decreases. This indicates that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters and acceleration factor in both cases of censoring.
3. The asymptotic variances of the estimators are decreases as sample size is increasing (Table 2 and Table 5).
4. The interval of the estimators decreases when the sample size is increasing. Also, it can be seen that the interval of the estimators at 95% confidence level is smaller than the interval of estimators at 99% confidence level (Table 3 and Table 6).

6. Conclusion

In this article, we considered the testing step-stress PALT for the Rayleigh distribution under Type-I and Type-II censored data. In step-stress PALT, the items are run at both normal and accelerated conditions. Under Type-I censoring, the test unit is first run at normal use condition, and if it does not fail for a specified time τ, then it is run at accelerated condition until censoring time Y_c is reached. While, in the case of Type-II censoring, the test unit is first run at normal use condition, and if it does not fail for a specified time τ, then it is run at accelerated condition until number of failures r is reached.

We present the performance of the ML method to estimate the Rayleigh parameter and acceleration factor in step-stress PALT for Type-I and Type-II censoring case. The simulation results show that the ML method performs well in most cases in terms of the MSE, RABias and RE. Hence, we could conclude that the second set of parameters have good statistical properties than the first set of the parameters for all sample sizes for Type-I and vise versa for Type-II censoring. It is also observed from numerical that MLE are consistent and asymptotically normally distributed. Thus, ML method is a good approach to estimate the parameters of the Rayleigh distribution and the accelerated factor in step-stress PALT under censored data.

The further research could be used other methods such as the expectation maximization (EM) algorithm to estimating the parameters of Rayleigh distribution and the accelerated factor in step-stress PALT under Type-I and Type-II and multiply censored data or progressive censoring.

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