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Orbit of a point in Dynamical Systems

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Abstract In this paper, we have proved the necessary and sufficient condition for a weakly mixing and topologically mixing function. Some properties of the monoid, periodic points and eventually periodic points are obtained. Some relations between weakly mixing, transitive and topologically mixing functions are obtained. Some results of considerable importance about the orbit of a point and relation with eventually periodic point are proved. Some results of the set theory that play an important role in our studies are included. Some new terms like singly transitive and lately transitive are introduced.

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1. INTRODUCTION

There are situations or sometimes there arises a situation where the things under consideration are not clear or are vague. Sometimes we come across a 'setup` where we are unable to say whether we are precise or not; at times we can clearly see that we are not precise. Similarly, sometimes the outcome of our considerations may not be certain. In short we can say that there is an uncertainty or chaos ([1]) in the situation.

So there arises a question: Can we handle these terms mathematically? Ideally speaking there should be no place for any one of these terms in the

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study of Mathematics. In fact Mathematics means precision. But there is need to take care of these things in a suitable way by an application of Mathematics. To think of a practical solution of these problems, there is need to use Mathematics i.e. there is need to take care of these things in a suitable way by an application of Mathematics. One approach in this direction is to develop a model. To do this we need some mathematical tools. Efforts have been done especially in the last about fifty years. Some new studies have been developed to help take care of these things with mathematical precision or with as much of mathematical precision as is possible. These new studies include: the theory of fractals ([2]) and dynamical systems ([3]).

By making use of knowledge of these mathematical concepts, we try to 'change` vague things or chaotic/uncertain things into precise things, rather we develop a way of study where vague things or uncertain things can be viewed as precisely as possible. Since we have to bring Mathematics in one form or the other, we may need a number of variables along with the other things. The numbers of variables depends upon the degree of complexity in the original version.

In this paper we shall consider only dynamical systems.

H. Poincare was the first who introduced the topological notions and methods in dynamics. It was initially for the study of ordinary differential equations. It was remarked by O.Frink ([4]) that many mathematical systems are at the same time lattices and topological spaces. Therefore, involvement of topological spaces/metric spaces can be helpful. G.D. Birkhoff ([3]) was the first who undertake the well ordered development of topological dynamics, indicating its fundamental notional character and making basic contributions. G.T. Whyburn's book ([6]) contains some material related to topological methods in dynamics. But most of the work regarding dynamical systems is done in the last about twenty years. The sensitivity, transitivity have been earlier studied e.g. see [5].

In section I, we study some preliminaries related to the composition of functions. In section 2, we study the properties of a monoid and orbit of a point of a function in a dynamical system. In section 3, we study the relationship of the weakly mixing, topologically mixing and transitive functions. First we give some definitions and notation that are needed herein.

2. NOTATION AND DEFINITIONS

Let X is a nonempty set. Let $f : X \rightarrow X$. By definition $f^0 = id_X$ i.e. identity function on X, $f^1 = f$ and for a positive integer $n \geq 2$, inductively, $f^n = f \circ f^{n-1}$. Similarly, for $x \in X$, $f^0(x) = x$, $f^1(x) = f(x)$ and for a positive integer $n \geq 2$, inductively, $f^n(x) = f(f^{n-1}(x))$.

For sets X and Y, let g, f : X → Y. The function **g x f** : X x X → Y x Y is defined as (g x f)(x, y) = (g(x), f(y)), for (x, y) ∈ X x X.

Binary operation on a set X is a function ∗ : X x X → X, for a, b ∈ X, we write a ∗ b for ∗(a, b). A binary operation ∗ is **associative**, if a ∗ (b ∗ c) = (a ∗ b) ∗ c, for all a, b, c ∈ X. A set X with a binary operation ∗ on it is called **semi group** if binary operation ∗ is associative. An element e of X is called **identity,** of binary operation ∗ if a ∗ e = a = e ∗ a, for all a ∈ X. The identity element whenever exists, is unique. The pair (X, ∗) is called a **monoid,** if ∗ is associative and there is an identity element e in X.

Let (X, ∗) be a monoid. For an element a of X, we define $a^0 = e$ and $a^1 = a$ and for a positive integer n ≥ 2, we define inductively, $a^n = a * a^{n-1}$.

A set X with a metric/topology on it is a **metric/topological space**. A collection b of subsets of X is called a base for open sets of X, if each member of b is open and every open set of X is union of some members of b.

Let X be a metric/topological space Let f : X → X. The pair (X, f) is called a **dynamical system,** if f is continuous. Let x be an element of X. The set $\{f^n(x) \mid n \geq 0\}$ is called the **orbit** of the point x. The orbit of x is denoted by **Orb(x)** or **Orb(x, f)** if there is need to mention f. A point x ∈ X is called **periodic** if $f^k(x) = x$ for some positive integer k. The smallest k such that $f^k(x) = x$ is called the **period of x**. A point x is called **eventually periodic** if for some integer t ≥ 0, $f^t(x)$ is periodic.

f : X → X is called **transitive** if for any pair of nonempty open sets G, V of X, there exists a positive integer n such that $f^n(G) \cap V \neq f$. If f^n is transitive for every positive integer n, then f is called **totally transitive**. f : X → X is called **weakly mixing** if for every pairs of nonempty open sets G_1, G_2 and V_1,V_2 of X , there exists a positive integer n, such that $f^n(G_i) \cap V_i \neq f$ for i = 1, 2. f is called **singly transitive** if f(A) ∩ B ≠ f for nonempty open sets A, B of X. f is called **lately transitive** if there exists a positive integer k, such that f^n is singly transitive for every positive integer n ≥ k. f is called **mixing** or **topologically mixing** if for each pair of nonempty open sets G, V in X, there exists a positive integer k such that $f^n(G) \cap V \neq f$ for all n ≥ k.

3. PRELIMINARIES

The following set-theoretic observations are added for ready reference.

Remark.1.1. Let X be a set. Let H, K ⊂ X then (i) H ∩ K = f iff H ⊂ X – K and K ⊂ X – H. (ii) H ∩ K ≠ f iff H Ë X – K and K Ë X – H.

Remark.1.2. Let f,g,h and k : X → X. Then (i) (h x g) o (k x f) = (h o k) x (g o f). (ii) $(g \times f)^n = g^n \times f^n$

Proof. Let $h \circ k, g \circ f : X \to X$. $h \times g, k \times f : X \times X \to X \times X$. $(h \times g) \circ (k \times f), (h \circ k) \times (g \circ f) : X \times X \to X \times X$. (i) Let $(x, y) \in X \times X$. $((h \times g) \circ (k \times f))(x, y) = (h \times g)((k \times f)(x, y)) = (h \times g)(k(x), f(y)) = (h(k(x)), g(f(y))) = ((h \circ k)(x), (g \circ f)(y)) = ((h \circ k) \times (g \circ f))(x, y)$. (ii) $(g \times f)^n, g^n \times f^n : X \times X \to X \times X$. The result holds for $n = 1$. Suppose the result holds for n. We prove it for $n + 1$. $(g \times f)^{n+1} = (g \times f)^n \circ (g \times f) = (g^n \circ g) \times (f^n \circ f) = g^{n+1} \times f^{n+1}$ by (i).

Remark.1.3. Let X, Y be two sets. Let $A, H \subset X$ and $B, K \subset Y$. Then $(A \times B) \cap (H \times K) = (A \cap H) \times (B \cap K)$.

Proof. Let $(x_1, x_2) \in (A \times B) \cap (H \times K) \Leftrightarrow (x_1, x_2) \in (A \times B)$ and $(x_1, x_2) \in (H \times K) \Leftrightarrow x_1 \in A, x_2 \in B$ and $x_1 \in H, x_2 \in K \Leftrightarrow x_1 \in A$ and $x_1 \in H$ and $x_2 \in B$ and $x_2 \in K \Leftrightarrow x_1 \in (A \cap H)$ and $x_2 \in (B \cap K) \Leftrightarrow (x_1, x_2) \in (A \cap H) \times (B \cap K)$. Hence $(A \times B) \cap (H \times K) = (A \cap H) \times (B \cap K)$.

Remark.1.4. Let $g, f : X \to X$ and $A, B \subset X$. Then (i) $(g \times f)(A \times B) = g(A) \times f(B)$. (ii) $((g \times f)^n (A \times B)) \cap (V \times H) = (g^n(A) \cap V) \times (f^n(B) \cap H)$.

Proof. (i) It follows using definition of $g \times f$. (ii) By Remark1.2(ii), $((g \times f)^n(A \times B)) \cap (V \times H) = ((g^n \times f^n)(A \times B)) \cap (V \times H) = (g^n(A) \times f^n(B)) \cap (V \times H)$, using (i). Applying Remark1.3, we get the result.

Remark.1.5. If X be metric/topological space, then $X \times X$ is a metric/topological space. Let $\beta = \{A \times B : A \text{ and } B \text{ open in } X\}$. Then β is a base for open sets of $X \times X$.

The following follows using definition of weakly mixing and transitive.

Remark.1.6. Let X be a metric/topological space. If $f : X \to X$ is weakly mixing, then f is transitive.

4. MONOID AND ORBIT OF A POINT

Remark.2.1. Let M be a monoid. For an element a of M and for positive integers n and m, (i) $a * a^n = a^n * a$. (ii) $a^n * a^m = a^{n+m}$.

Proof. (i) Using induction method, for $n = 1$, the result holds. Now suppose, the result is true for $n - 1$. Now $a * a^n = a * (a * a^{n-1}) = a * (a^{n-1} * a) = (a * a^{n-1}) * a = a^n * a$. (ii) If $n = 1$, $a * a^m = a^{m+1} = a^{1+m}$. Suppose the result is true for $n - 1$. Take $a^n * a^m = (a * a^{n-1}) * a^m = a * (a^{n-1} * a^m) = a * a^{n+m-1} = a^{n+m}$.

Let X be a nonempty set. Let M(X) be the collection of all self maps on X, i.e. $M(X) = \{f \mid f : X \to X\}$. For $x \in X$, let $M_x = \{f^n(x) \mid n \geq 0\}$.

We shall need the following Remark. We may use it without mentioning it.

Remark.2.2. (i) M(X) is a monoid where the binary operation $*$ is the composition of functions. (ii) Let $x \in X$. For $f^s(x), f^q(x) \in M_x$, we define $f^s(x) * f^q(x) = f^{s+q}(x)$. M_x becomes a monoid. (iii) Let $x \in X$. In view of Remark2.1(ii), for positive integers n and m, $f^{n+m}(x) = f^n(f^m(x))$.

Remark.2.3. Let k be the period of $x \in X$, then (i) $f^{mk}(x) = x$ for every positive integer m. (ii) If $m > k$ and $f^m(x) = x$, then $m = kq$, for some $q \in IN$. (iii) For $n \in IN$, $f^n(x) = f^r(x)$, where r is a nonnegative integer r with, $r < k$. (iv) Orb(x) $= \{x, f(x), \ldots\ldots\ldots\ldots, f^{k-1}(x)\}$. (v) For $0 < r, s < k$, if $f^s(x) = f^r(x)$, then $r = s$. (vi) Orb(x) contains exactly k elements. (vii) Let $0 < r < k < s$, then $s = r + kq$ for some positive integer q.

Proof. As k is the period of x, therefore $f^k(x) = x$. (i) We use Induction on m. For $m = 1$, $f^k(x) = x$. Suppose the result is true for m-1. Now $f^{mk}(x) = f^{(m-1+1)k}(x) = f^{(m-1)k+k}(x) = f^k(f^{(m-1)k}(x)) = f^k(f(x)) = f^k(x) = x$, using Remark2.2(iii) and result for m-1. (ii) We can write $m = kq + r$, for q and r in IN with $0 \leq r < k$. $f^m(x) = f^{kq+r}(x) = f^r(f^{kq}(x)) = f^r(x)$ by using (i) and Remark2.2(iii). Therefore, $f^r(x) = x$. But this is contradiction, unless $r = 0$, as $r < k$. (iii) Let $n \in IN$. We can write $n = mk + r$, for some m and r in IN with $0 \leq r < k$. $f^n(x) = f^{mk+r}(x) = f^r(f^{km}(x)) = f^r(x)$ by (i) and Remark2.3(iii). (iv) By (iii), for $n \in IN$, $f^n(x) \in$ Orb(x) as $n < k$. (v) We suppose $r > s$. Since $k - r > 0$, using Remark2.2(iii), $f^{k-r+s}(x) = f^{k-r}(f^s(x)) = f^{k-r}(f^r(x)) = f^k(x) = x$. Therefore, $r = s$, otherwise $k - r + s < k$, which is not possible. (vi) In view of (v) and (iii), Orb(x) contains exactly k elements. (vii) As $0 < r < k$, and $s \neq k$. We can write $s = qk + t$, for some $0 < t < k$. By (i), $f^{qk}(x) = x$. Therefore, $f^t(x) = f^t(f^{qk}(x)) = f^{t+qk}(x) = f^s(x) = f^r(x)$ by (v). Thus $t = r$.

Lemma.2.4. Let $x \in X$ be eventually periodic. There exists an integer $t \geq 0$ such that $f^t(x)$ is periodic. Let k be the period of $f^t(x)$. Then (i) $f^{k+t}(x) = f^t(x)$. (ii) $f^{k+t+r}(x) = f^{t+r}(x)$ for all $r \geq 0$. (iii) $f^{2k+t-1}(x) = f^{k+t-1}(x)$. (iv) $f^{mk+t-1}(x) = f^{k+t-1}(x)$ for all $m \geq 1$. (v) $f^{m(k+t)}(x) = f^{mt}(x)$. (vi) $f^{2k+t-r}(x) = f^{k+t-r}(x)$ for all $r \leq k$. (vii) $f^{mk+t-r}(x) = f^{k+t-r}(x)$ for all $r \leq k$ and for all $m \geq 1$.

Proof. (i) Let $f^t(x)$ be periodic with period k. So with the help of Remark2.2(ii), $f^{k+t}(x) = f^k(f^t(x)) = f^t(x)$. (ii) Using Remark2.2(ii), $f^{t+k+r} = f^r o f^{k+t}$. So $f^{t+k+r}(x) = f^r(f^{k+t}(x)) = f^r(f^t(x)) = f^{t+r}(x)$. (iii) If we take $r = k-1$ in (ii), we get $f^{k+t+k-1}(x) = f^{t+2k-1}(x) = f^{t+k-1}(x)$. (iv) We use Induction on m. For $m = 1$, it is obvious. Suppose the result is true for m-1. Then $f^{(m-1)k+t-1}(x) = f^{k+t-1}(x)$. $f^{mk+t-1}(x) = f^{(m-1+1)k+t-1}(x) = f^{(m-1)(k+t-1)+k}(x) = f^k(f^{(m-1)k+t-1}(x)) = f^k(f^{k+t-1}(x)) = f^{2k+t-1}(x) = f^{k+t-1}(x)$, using (ii) and Remark2.2(ii). (v) We use induction on m. For $m = 1$, the result is obvious. Suppose the result is true for m-1, i.e. $f^{(m-1)(k+t)}(x) = f^{(m-1)t}(x)$. Now $f^{m(k+t)}(x) = f^{(m-1+1)(k+t)}(x) = f^{(m-1)(k+t)+(k+t)}(x) = f^{(k+t)}(f^{(m-1)(k+t)}(x)) = f^{(k+t)}(f^{(m-1)t}(x)) = f^{(m-1)t}(f^{(k+t)}(x)) = f^{(m-1)t}(f^t(x)) = f^{mt}(x)$ by (i). (vi) Taking k-r in place of r in (ii), we get $f^{t+k-r}(x)$

$= f^{k+t+k-r}(x) = f^{2k+t-r}(x)$. (vii) We use induction on m. For m = 1, it is obvious. Suppose the result is true for m-1. $f^{mk+t-r}(x) = f^{(m-1+1)k+t-r}(x) = f^k(f^{(m-1)k+t-r}(x)) = f^k(f^{k+t-r}(x)) = f^{2k+t-r}(x) = f^{k+t-r}(x)$ by (vi).

Proposition.2.4. The orbit of a point x is finite iff it is eventually periodic.

Proof. $Orb(x) = \{f^n(x) \mid n \geq 0\}$ is the orbit of x. First suppose that Orb(x) is finite. Therefore there exists a positive integer m such that $Orb(x) = \{f^0(x), f^1(x), f^2(x) \ldots\ldots\ldots\ldots, f^m(x)\}$. Let $r \in IN$. Since $f^{m+r}(x) \in Orb(x)$, there exists t_r, $0 \leq t_r \leq m$ such that $f^{m+r}(x) = f^{t_r}(x)$. Let $k = m + r - t_r$. Since $m - t_r \geq 0$ therefore $k = m + r - t_r > 0$. $f^k(f^{t_r}(x)) = f^{k+t_r}(x) = f^{m+r}(x) = f^{t_r}(x)$. Thus $f^{t_r}(x)$ is periodic. Therefore, x is eventually periodic. Now suppose that x is eventually periodic. Then for some integer $t \geq 0$, $f^t(x)$ is periodic. If t = 0, then x is periodic. Therefore Orb(x) is finite. So we suppose that $t \geq 1$. Let k be the period of $f^t(x)$. We claim that $Orb(x) = \{f^0(x), f^1(x), f^2(x) \ldots\ldots\ldots\ldots, f^{k+t-1}(x)\}$. Otherewise $n > k + t$, so we can write $n - t = mk + r$ for some $m \in IN$ and a nonnegative integer and r with $0 \leq r < k$. We have $f^n(x) = f^{mk+t+r}(x) = f^{t+r}(f^{mk}(x)) = f^{t+r}(x)$, using Remark2.3(i). So $f^n(x) \in \{f^0(x), f^1(x), f^2(x) \ldots\ldots\ldots\ldots, f^{k+t-1}(x)\}$ as $t + r < k + t - 1$.

Remark.2.5. In the proof of the above result we get a description of the orbit of every eventually periodic point of the space. Also we have obtained, for a point x with finite orbit, possibly all $f^t(x)$ which are periodic points.

Remark.2.6. Let $x \in X$ be eventually periodic. For $f^s(x), f^q(x) \in Orb(x)$, we define $f^s(x) * f^q(x) = f^{s+q}(x)$. Then Orb(x) is a monoid.

Proof. By Proposition2.4, $Orb(x) = \{f^0(x), f^1(x), f^2(x), \ldots\ldots\ldots\ldots\ldots\ldots f^{k+t-1}(x)\}$. Since $f^{s+q}(x) \in Orb(x)$, Orb(x) is a monoid in view of Remark2.2(ii).

5. SOME NEW OBSERVATIONS

Let X be a metric/topological space. By Remark1.5, X x X is a metric/ topological space. For g, f : X → X, we find some relations between g, f, f x f and g x f from the point of view of transitivity etc.

Remark3.1. Let X be a metric/topological space. Let β be a base for open sets of X. Let f : X → X. (i) f is transitive if every pair A, B of nonempty members of β, there exists a positive integer n such that $f^n(A) \cap B \neq \phi$. (ii) f is weakly mixing if for every pairs of nonempty members G_1, G_2 and V_1,V_2 in X , there exists a positive integer n, such that $f^n(G_i) \cap V_i \neq \phi$ for i = 1, 2.

Proof. (i) Let G and V be nonempty open sets of X. Using definition of open base, we find nonempty members A and B of β such that $A \subset G$ and $B \subset V$. As there exists a positive integer n such that $f^n(A) \cap B \neq \phi$, we have $f^n(G) \cap V \neq \phi$. (ii) Let G_1, G_2 and V_1, V_2 be pairs of nonempty open sets of X . Using definition of open base, there exists pairs of nonempty members A_1, A_2 and B_1, B_2 of β such that $A_i \subset G_i$ and $B_i \subset V_i$, for i = 1, 2. By the given condition, there exists a positive integer n such that $f^n(A_i) \cap B_i \neq \phi$ for i = 1, 2. This implies that $f^n(G_i) \cap V_i \neq \phi$ for i = 1, 2.

Proposition.3.2. Let g, f : X → X. (i) g x f is transitive iff f x g is transitive. (ii) g x f is weakly mixing iff f x g is weakly mixing.

Proof. We shall use Remark3.1. By Remark1.5, β = {A x B : A and B open in X} is a base for open sets of X x X. (i) Suppose g x f is transitive. To prove f x g is transitive, let A x B and V x H be nonempty members of β. Then B x A and H x V are nonempty open sets of X x X. Since g x f is transitive, a positive integer n exists such that $((g \times f)^n(B \times A)) \cap H \times V \neq \phi$. By the Use of Remark1.4, $((g \times f)^n (B \times A)) \cap H \times V = (g^n(B) \cap H \times f^n(A) \cap V)$. Therefore $(g^n(B) \cap H \neq \phi$ and $f^n(A) \cap V) \neq \phi$. Thus $(f^n(A) \cap V)) \times ((g^n(B) \cap H) \neq \phi$. Now using Remark1.4, $((f \times g)^n(A \times B)) \cap V \times H \neq \phi$. The converse follows by interchanging g and f. (ii) Suppose g x f is weakly mixing. To prove f x g is weakly mixing, let $(A \times B)_i$ and $(V \times H)_i$ be nonempty members of β for i = 1, 2. Then $(B \times A)_i$ and $(H \times V)_i$ are nonempty open sets of X x X for i = 1, 2. Since g x f is weakly mixing, there exists a positive integer n such that $((g \times f)^n(B \times A)_i) \cap (H \times V)_i \neq \phi$, for i = 1, 2. Proceeding as in part (i), $((f \times g)^n(A \times B)_i) \cap (V \times H)_i \neq \phi$, for i = 1, 2. The converse follows by interchanging g and f.

Proposition.3.3. Let g and f : X → X. If g x f is transitive, then g and f are (i) weakly mixing (ii) transitive.

Proof. g x f : X x X → X x X. Let A, B and V, H be two pairs of nonempty open sets of X. A x B and V x H are nonempty open sets of X x X. Since g x f is transitive, there exists a positive integer n, such that $((g \times f)^n(A \times B)) \cap V \times H \neq \phi$. Using Remark1.4, $((g \times f)^n (A \times B)) \cap V \times H = (g^n(A) \cap V \times f^n(B) \cap H)$. Therefore $(g^n(A) \cap V \neq \phi$ and $f^n(B) \cap H) \neq \phi$. Similarly, since B x A and H x V are nonempty open sets of X.xX, we have $(g^n(B) \cap H \neq \phi$ and $f^n(A) \cap V) \neq \phi$. This proves that g and f are weakly mixing. (ii) It follows by (i) and Remark1.6.

Proposition.3.4. f x f is transitive iff f is weakly mixing.

Proof. Let f x f be transitive, using Proposition3.3(i), f is weakly mixing. Now suppose that f is weakly mixing. For proving f x f to be transitive, we shall use here Remark3.1. Now by Remark1.5, $\beta = \{A_1 \times A_2 : A_i$ open in X for i = 1, 2} is a base for open sets of X x X. Let $A_1 \times A_2$ and $V_1 \times V_2$ be nonempty members of β. Then, for i = 1, 2, A_i and V_i, are two pairs of nonempty open sets of X. Since f is weakly mixing, there exists a positive integer n such that $f^n(A_i) \cap V_i)$

$\neq \phi$ for i = 1, 2. This implies that $(f^n(A_1) \cap V_1)) \times (f^n(A_2) \cap V_2) \neq \phi$. But, by Remark 1.4(ii), $((f^n(A_1) \cap V_1) \times (f^n(A_2) \cap V_2) = ((f \times f)^n(A_1 \times A_2)) \cap V_1 \times V_2$. Therefore, $((f \times f)^n(A_1 \times A_2)) \cap V_1 \times V_2 \neq \phi$.

For a metric/topological space X, let $f : X \to X$. For a positive integer n, denote fxfx……..xf (n-times) by f^{xn}.

Proposition3.5. (i) If f is weakly mixing, then f^{xn} is transitive for every positive integer n. (ii) If f^{xk} is transitive for some positive integer $k > 1$, then f is weakly mixing. (iii) If f^{xk} is transitive for some positive integer $k > 1$, then f^{xn} is transitive for every positive integer n.

Proof. (i) If n = 1, the result follows by Remark1.6. If we take n = 2, then by reverse part of Proposition3.4, f x f is transitive. For $n > 2$, the proof is obtained on the lines of proof of f x f. (ii) Since $k > 1$, $f^{xk} = g \times f$, where $g = f^{x(k-1)}$. Now, by Proposition3.3(i), f is weakly mixing. (iii) It follows using (ii) and (i).

Remark 3.6. f is lately transitive iff f is topologically mixing.

Proof. Suppose f is lately transitive. Then there exists a positive integer k for which f^n is singly transitive for every $n \geq k$. Let G and V be nonempty open sets of X. Let $n \geq k$. Since f^n is singly transitive, $f^n(G) \cap V \neq \phi$. Thus f is topologically mixing. The converse follows on the same lines.

CONCLUSION

In section 2 of this paper, we have studied the period, eventually periodic point, orbit of a point and properties of eventually periodic point for the monoid M(X). We have proved the relation between orbit of a point and eventually periodic point for the monoid M(X). In section 3, we have studied that weakly mixing and transitive functions are commutative processes. We have also proved the relation between transitive, weakly mixing, lately mixing and topologically mixing. The results obtained above are expected to be used for further study of orbits of points and transitivity of function in some dynamical systems.

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