INVESTIGATION OF REGULAR & CHAOTIC BEHAVIOUR IN SOME BIOLOGICALLY INTERACTIVE MODELS

THESIS

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DOCTOR OF PHILOSOPHY

By

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DECLARATION BY THE STUDENT

I hereby certify that the work which is being presented in this thesis entitled **"Investigation of regular & chaotic behavior in some biologically interactive models"** is for fulfilment of the requirement for the award of degree of Doctor of Philosophy submitted in the Department of Applied Sciences, Chitkara University, Himachal Pradesh is an authentic record of my own work carried out under the joint supervision of **Dr L M Saha** and **Dr Ashok K Chitkara.**

The work has not formed the basis for the award of any other degree or diploma, in this or any other Institution or University. In keeping with the ethical practice in reporting scientific information, due acknowledgements have been made wherever the findings of others have been cited.

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CERTIFICATE BY THE SUPERVISORS

This is to certify that the thesis entitled **"Investigation of regular & chaotic behaviour in some biologically interactive models" submitted** by **Ms Rajni Sharma with Regd. No. PHDASM09015** to the Chitkara University, Himachal Pradesh in fulfilment for the award of the degree of **Doctor of Philosophy** is a *bonafide* record of research work carried out by her under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institution or University for the award of any degree or diploma.

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ABBREVIATIONS

LCE	Lyapunov characteristic exponent
FLI	Fast Lyapunov indicator
SALI	Smaller alignment indicator
DLI	Dynamic Lyapunov indicator

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ABSTRACT

This thesis provides an examination of chaotic and regular patterns in some biologically interactive models through the conceptual lens of chaos theory. Chaos theory is chosen as an analytical tool because it allows us to reveal the patterns and processes of complex systems as they move between order and disorder.

The central question is that of how complexity, which is based in chaos theory, can highlight the ways to address macro and micro level problems in complex biological systems.

Four biological systems are discussed in terms of complexity. The first system under study was a three-tropic food chain model that has been investigated by discretizing the classical mathematical model for three-trophic food chains modified and used by Bo Deng (2006). Famous Euler's method has been employed to discretize the differential equations used in the work of Bo Deng. Various measurable quantities for emergence of chaos, like Lyapunov exponents, topological entropies, correlation dimensions, have been numerically calculated and represented through plots. Finally, the chaos indicator, named Dynamic Lyapunov Indicator (DLI), has been used to identify clearly chaotic and regular motion. Bifurcation diagrams and various plots for LCEs, topological entropies, correlation dimension etc. are interesting and provide help to properly analyze the evolutionary behavior.

Another problem on dynamics of two-gene Andrecut-Kauffman system has been studied for chaos and complexity. The model consists of nonlinear equations, in context with biochemical phenomena obtained from chemical reactions appearing in a two-gene model (Andrecut and Kaufmann, 2007). The chemical reactions are assumed to correspond to gene expression and regulation. For this problem, studies have been performed carefully to understand chaotic phenomena during its evolution together with complexities present in the system.

The third problem is based upon complexities in a Plant-Herbivore system. The studied work is based on the non-dimensional mathematical model proposed in a recent article (Abbott and Dwyer, 2007). Mathematical analysis and simulations of this model provide us with biological insights that may be used to devise control strategies to regulate the population of the herbivore. Since the herbivore movements are random, it is more appropriate to study a stochastic model instead of deterministic one. Such realistic plant-herbivore system, would be our future aim of investigation.

Lastly, we have worked with a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period. The measures, like Lyapunov exponents, topological entropies and correlation dimensions, are obtained for this problem for discussion of evolutionary phenomena. In the processes of study, we have discussed the stability criteria of the steady state solution.

It is concluded that complexity, based on chaos theory, is a powerful framework for understanding the interactions in biological systems. Chaos theory provides us with many tools which can indicate regular and turbulent patterns which could be of great use if we wish to make some changes in the existing systems. This can answer various questions regarding existence and extinction of various species or the impact of their social interactions on the environment.

Chapter 1

Introduction

The things that really change the world, according to Chaos theory, are the tiny things. A butterfly flaps it's wings in the Amazonian jungle, subsequently a storm ravages half of the Europe.
Neil Gaiman

1.1 Brief account of research problem

Chaos theory comes within the domain of nonlinear systems. Chaos in a system signifies a state of the system that shows unpredictability. Prediction is almost impossible at this state since the system becomes very sensitive to the initial condition once it is chaotic. Almost all natural and biological systems are nonlinear in nature and many of them are full of complexity and disorder. Nonlinear systems do not obey the principle of superposition, and so there cannot be a simple rule that can describe all nonlinear systems. However, the nature is a mixed manifestation of order and disorder and its beauty lies in such manifestation. For a certain set of values of parameters, a system shows regular behaviour; the same system may suddenly show chaos for another set of values of parameters. Evolutionary behaviours and properties of two nonlinear systems could be completely different. Since these systems are part of our life, we need to explore them and investigate their properties by proposing specific rules. These days, we have high speed computers and software of various levels and purpose. With the knowledge of our mathematics and basic sciences, if we can write appropriate mathematical models for any natural system then with the aid of computers we can explore various properties of the system that emerge during evolution. The system proposed may incorporate some parameters and, at a certain stage, set of parameter values may lead the system into chaos. In some sense, chaos is the property of "Nonlinearity" of the system which may happen very often in any such system. Thus, we observe:

chaos in weather like cyclone, hurricane, tsunami, tornado etc; chaos in society like spread of epidemic, communal riots, population explosion etc; chaos in markets like unpredictable fluctuation of economy; chaos in medical sciences like erratic blood flows, complex structure of our brain, heart attack etc. Chaos theory is applied in many scientific disciplines, including geology, mathematics, microbiology, biology, computer science, economics, engineering, finance, meteorology, philosophy, physics, politics, population dynamics, psychology, and robotics. Chaotic behaviour has been observed in the laboratory in a variety of systems, including electrical circuits, lasers, oscillating chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices, as well as computer models of chaotic processes. Observations of chaotic behaviour in nature include changes in weather, the dynamics of satellites in the solar system, the time evolution of the magnetic field of celestial bodies, population growth in ecology, the dynamics of the action potentials in neurons, and molecular vibrations.

To study nonlinear systems systematically, we need to use principles of nonlinear dynamics and various tools which are developed during the processes of investigations of different systems. A number of methods and tools are already available through results of a number of recent researches. To identify regular and chaotic motion in nonlinear systems presently we have tools like time series graphs, phase plots, bifurcation diagrams, Poincare maps and Poincare surface of section, Power spectrum analysis, bifurcation analysis, Lyapunov exponents and more recently, fast Lyapunov Indicators (FLI), Smaller Alignment Indices (SALI), and Dynamic Lyapunov Indicator (DLI). For certain systems we may also use some advanced concepts like topological entropy and correlation dimensions. Lyapunov exponents, topological entropy and correlation dimensions are considered as measure of chaos as well as complexities in nonlinear complex dynamical systems.

1.2 Importance of the area of research

In the proposed thesis, we have investigated the evolutionary behaviour in some biological and natural systems. Our models are both discrete and continuous. Existence of non-linear models is close to the natural phenomena happening around us. Such systems are full of complexity and may evolve chaotically. By studying the dynamics of such models, one can understand the evolutionary behaviour of that particular system e.g. studying an economic model on market, fluctuation and chaotic changes occurring there can be explained mathematically. Similarly studying a population model can throw light on the stability of population as well as survival under various conditions e.g. an epidemic and provides us some indications about preventive measures. With migration of population to bigger cities and other socio-economic changes, there are many changes in the weather and environment. We know that the weather is not the same everywhere nor it is same every day. Weather change is nothing but the phenomena of evolution of weather and it comes under the domain of nonlinear dynamics. A chaotic weather e.g. Tsunami, Tornado, Hurricane etc are due to change of certain parameters in the weather. Lorenz 1963, had proved that a small flip of butterfly can result into some devastating events such as hurricane, cyclone etc., at some larger distance.

Our environment is really vast and highly complex. The diversity of biological species and the complexity of every individual organism has a great role in this complexity. Ecosystems throughout the world are going through dramatic changes in their environment. This has resulted in the extinction of some species over all disturbing the system.

The individuals interact by predation, competition or cooperation on a smaller scale but on a larger scale, the microscopic interactions lead to interactions between populations. Food webs, which are networks of predator-prey interactions, provide a basic understanding of ecosystems and studying these in detail can help to find strategies to preserve the extinct species.

In population dynamics, one can easily observe invasions, population bursts and extinctions. These dynamics are the driving forces behind biological evolution. The emergence and extinction of species happens in a complicated manner and provides a basis for some exciting studies.

Human population is growing and so are the human impacts on the environment such as habitat destruction and fragmentation, climate change, species invasions, pollution, and overfishing. Due to this, many of the earth's ecosystems are experiencing large species losses. These losses raise many questions. For example, the extent to which the loss of biodiversity matters or what will happen to the productivity and balance of the ecosystems? These questions are very tough to answer due to the fact that ecological communities are very complex in themselves because of a lot of inter species interactions and interactions with the environment. Since biological succession always takes place, it is not possible to conserve every single species. Still, major changes that involve the destruction of whole ecosystems have to be taken care of. We cannot stop all human activities that impact environment but we can try to find efficient strategies for the conservation of the ecosystems. In order to do that, a deep and clear understanding of ecosystem functioning is required.

1.3 Literature review

An important area of research is the study of spread of epidemic in our social systems. In this regard, the Kermac-McKendric SIR model was proposed by Kermac and McKendric (Kermack and McKendrick, 1927). The SIR model is one of the simplest compartmental models where S denotes the number of susceptibles, I denotes the number of infections and R denotes the number of recoveries. The Kermack-McKendrick model is an SIR model for the number of people infected with a contagious illness in a closed population over time. It was proposed to explain the rapid rise and fall in the number of infected patients observed in epidemics such as the plague (London 1665-1666, Bombay 1906) and cholera (London 1865). More complicated versions of the Kermack-McKendrick model that better reflect the actual biology of a given disease are often used.

Stephen Smale (Smale, 1967) made remarkable contributions to the area of nonlinear dynamics. His first contribution was the Smale horseshoe that led to significant research in dynamical systems. To investigate biological models some works were reviewed among which the article of May (May et al., 1981) is very important which provides simple mathematical models for very complex dynamics. The American physicist Mitchell Jay Feigenbaum (Feigenbaum, 1978) laid the foundations for studying the world of complicated events in nature by recognizing patterns underlying the application of mathematical equations. Chirikov (Chirikov, 1979) worked on the universal instability of many-dimensional oscillator system.

Grassberger and Procaccia (Grassberger and Procaccia, 1983a, Grassberger and Procaccia, 1983b) worked on measuring the strangeness of Strange Attractors. The book by Moon (Moon, 1987) discusses advanced research developments in chaotic dynamical systems accessible to undergraduate and graduate mathematics students as well as researchers in other disciplines. The Tribolium Model (a model describing the population growth in the flour beetle *Tribolium*) proposed by Costantino et al (Costantino et al., 1997) is an interesting problem to study by imposing some modification. The book by Bailey (Bailey, 1975) also provides some basic foundations in this field.

The book by Haefner (Haefner, 1996) led to some direction while proposing the model. The model proposed provides strong empirical evidence for deterministic mechanism complexity and aperiodicity in different species of population. Then, one can proceed for further study of following recent works by Steriloff et al (Strelioff and Hubler, 2006) and others.

The dynamics of biological systems under different ecological conditions have been investigated from time to time by different researchers i.e. (Holling, 1965), (Freedman and So, 1985, (Hastings, 1991 #42, Klebanoff and Hastings, 1994), (Deng, 2001), (Aziz-Alaoui et al., 2001), (Brin and Stuck, 2002) and many others. Some of these models are used as a basis for the work presented in this thesis. The recent work by Ivanchikov and Nedorezov in 2011, has suggested that the discretized form of models provides a more interesting and detailed findings. We have also used discrete form of models to carry out our research.

1.4 Basic concepts and definitions

In this section, we have briefly defined basic concepts used in the research work.. The purpose of this section is to provide a conceptual framework. Important points like the question of stability and complexity are discussed elaborately below.

1.4.1 Dynamical systems

Mathematically, a dynamical system can be defined as a set of prescriptions that determine the time evolution of a set of state variables. It refers to a system in which a function describes the time dependence of a point in a geometrical space. The study of dynamical systems is the focus of dynamical systems theory, which has applications to a wide variety of fields such as mathematics, physics, biology, chemistry, engineering economics and medicine. Dynamical systems are a fundamental part of chaos theory, logistic map dynamics, bifurcation theory, the self-assembly process, and the edge of chaos concept. The concept of a dynamical system has its origins in Newtonian mechanics.

A proper definition of dynamical system can be written as follows:

***Definition*: A dynamical system is a system which evolves with time from a prescribed initial condition (s) with a well defined rule (s).**

Thus, there are two main ingredients of dynamical systems:

(1) The prescribed initial condition(s), say x_0 which indicates from where the system started evolving .

(2) The fixed rule(s), say f, which indicates how the system is evolving.

The rule (s), f, must be written in the form of mathematical equation (s) e.g. discrete forms of equations, differential equations, integral equations, algebraic equations etc.

In our study, we are mainly concerned with the following types of dynamical systems:

a) Discrete Dynamical Systems where time is an integer variable
b) Continuous Dynamical Systems where time is a continuous variable.

1.4.2 Discrete dynamical systems

General forms of discrete dynamical systems are as follows:

$$X_{n+1} = f(X_n)$$

where $f: M \to M$, is a map of a state space into itself and X_n denotes the state at the discrete time n. The sequence $f\{X_n\}$ obtained by iterating the above equation starting from an initial condition X_0 is called the orbit of X_0 , with $n \in N$ or $n \in Z$, depending on whether or not map f is invertible.

Discrete one dimensional systems be represented by equations of the type
$x_{n+1} = f(x_n)$.

All functions of one variable representing a system are covered under one dimensional discrete dynamical systems.

Examples:

(1) $f(x) = a x^2 + b x + c$ is represented as $x_{n+1} = a x_n^2 + b x_n + c$.

(2) Logistics map $f(x) = \lambda x (1 - x)$ is represented as $x_{n+1} = \lambda x_n (1 - x_n)$.

(3) Epidemic Model $f(x) = a x^2 - 1$ is represented as $x_{n+1} = k x_n^2 - 1$.

Discrete two dimensional systems be represented by pair of equations of the type:
$x_{n+1} = f(x_n , y_n)$
$y_{n+1} = g(x_n , y_n)$
All pair of functions of two variables representing a system are covered under two dimensional discrete dynamical systems.

Examples:

Hénon map: $f(x, y) = 1 - ax^2 + y$, $g(x,y) = bx$ is represented as

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n \\ y_{n+1} = b y_n \end{cases}$$

A two-species population model is described by the discrete system

$$x_{n+1} = ax_n(1 - x_n - y_n)$$
$$y_{n+1} = bx_n y_n$$

Discrete three dimensional systems are represented by three equations of the type:
$x_{n+1} = f(x_n , y_n, z_n)$
$y_{n+1} = g(x_n , y_n, z_n)$
$z_{n+1} = g(x_n , y_n, z_n)$

All sets of three coupled equations involving three functions of three variables x, y, z and representing a dynamical system are represented by three discrete forms of equations.

Example: A discrete 3-dimensional food-chain model proposed by Elsadany, (2012), to study the ecosystems of three interacting species, each with non-overlapping generations can be given as

$f(x, y, z) = a x (1 - x) - b x y,$

$g(x, y, z) = c x y - d y z,$

$h(x, y, z) = r y z$

A discrete form of above model be given by

$x_n + 1 = a x_n (1-x_n)-b x_n y_n$

$y_n + 1 = c x_n y_n -d y_n z_n$

$z_n +1 = r y_n z_n$

1.4.3 Continuous dynamical system

A continuous dynamical system is represented by

$$\frac{dx}{dt} = f(x,t),\ x_0 \in X$$

where $X \in R^n$. So, starting from initial condition x_0 we can solve the equations to obtain the future state $x(t)$ of the given system and the path is called phase space as it evolves with time.

Continuous dynamical systems are represented by differential equations.

Examples:

One dimensional logistic equation $= \lambda x (1 - x)$.

2-Dimensional Predator-Prey system.

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = dxy - cy$$

Duffing Equation $\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta y + \mu y^3 = \lambda \cos \omega t$

The Lorenz system is a 3- dimensional continuous system which was introduced as a model for atmospheric convection by (Lorenz, 1963). It is governed by the following system of equations:

$$\frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{r(y - x)}$$
$$\frac{\mathbf{dy}}{\mathbf{dt}} = \mathbf{x(a - z)}$$
$$\frac{\mathbf{dz}}{\mathbf{dt}} = \mathbf{xy - bz}$$

where r is called the Prandtl number and a is called the Rayleigh number and b is the geometric factor.

1.4.4 Linear and nonlinear dynamical systems

Linear dynamical systems are dynamical systems whose evaluation functions are linear. Linear systems can also be used to understand the qualitative behaviour of general dynamical systems, by calculating the equilibrium points of the system and approximating it as a linear system around each such point.

A linear system obeys the Principle of superposition which can be described as follows:

Let **L** be any differential operator, **u**, **v** are two vectors from a vector space **V** and a, b are two scalars then if

L (a **u** + b **v**) = a **L** (**u**) + b **L** (**v**),

we say **L** is a linear differential operator.

The principle of superposition holds in linear systems while in a nonlinear system the change of the output is not proportional to the change of the input. These systems respond disproportionately (nonlinearly) to initial conditions or perturbing stimuli.

The differences between a linear and a non-linear system is that a linear system always responds by vibrating at the same frequency as the input. A non-linear system does not usually or necessarily respond at the same frequency as the input. We can understand this better by looking at the motion of a damped, forced non-linear pendulum equation.

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0{}^2 \sin\theta = f\cos\omega t$$

And the linear form is

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0{}^2\theta = f\cos\omega t$$

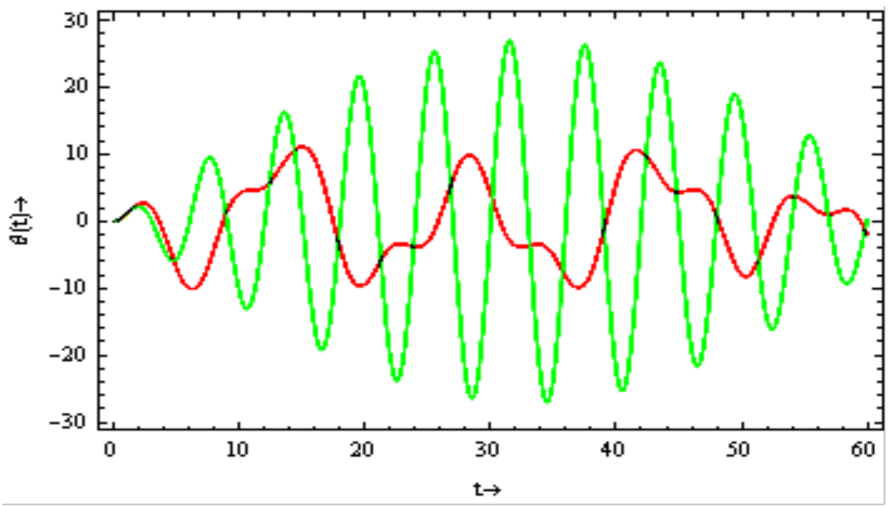


Fig.1.1: ***Non-linear (Red) and linear (Green) oscillations in equation of the pendulum.***

1.4.5 Deterministic and Stochastic Systems

(a) Deterministic Systems:

In mathematics and physics, a deterministic system is a system in which no randomness is involved in the development of future states of the system. A deterministic model will thus always produce the same output from a given starting condition or initial state.

If the rule, the mathematical equations and the initial conditions are well explained and no randomness is involved i.e. the output will be same for same input, then such a system is called deterministic. It is very important to note that in a deterministic system one gets the same output for any repeated same output.

(b) Stochastic (or Random) Systems:

In a Stochastic or a Random system, one gets different outputs for any repeated same input. An example of stochastic system is the random noise.

1.4.6 Fixed Points

Fixed point theory has been used to deal with stability problems for several years. Since here we are dealing with non-linear systems so the fixed points could be of any type such as stable & unstable node, saddle point, stable & unstable focus etc. If the dimension is greater than two, fixed points have to be categorised carefully. The Jacobian matrix for the system plays a significant role while finding fixed points and their stability.

To determine the nature of fixed points, it is important to learn about the controlling parameters in a system. A stable fixed point has a stable orbit initiated by it while an orbit initiated nearby an unstable fixed point is unstable. Clarity of fixed points leads to a clear understanding of evolutionary behaviour of a system.

Definition: **Fixed points are the steady state real solutions of a dynamical system. Fixed points are also known as equilibrium points, critical points, liberation points, Lagrangian points.**

A fixed point can be categorised as stable, unstable and semi-stable. A fixed point is said to be either stable or unstable depending on the behaviour of the trajectories in its neighbourhood. If all these trajectories remain near the fixed point, then the point is considered to be stable, and if any of these trajectories do not remain in a neighbourhood of the fixed point, the fixed point is considered unstable.

(a) Fixed points for discrete systems

One dimensional discrete system: $x_{n+1} = f(x_n)$

To find fixed point put

$x_{n+1} = x_n = x = f(x)$

Then solve the resulting equation for x. Only real solutions of the equation will be the termed as fixed points of the given system.

Example: Consider the map

$x_{n+1} = a - x_n^2$ or $f(x) = a - x^2$

For fixed point, we have $x = a - x^2$, which implies $x = \frac{-1}{2} \pm \frac{\sqrt{1+4a}}{2} \Rightarrow$ This system has fixed points $x_1^* = \frac{-1}{2} + \frac{\sqrt{1+4a}}{2}$ and $x_2^* = \frac{-1}{2} - \frac{\sqrt{1+4a}}{2}$ provided $a > -\frac{1}{4}$ and if $a = -\frac{1}{4}$, then the system has only one fixed point $x^* = \frac{-1}{2}$

Example: Consider the map

$x_{n+1} = x_n - x_n^2 - 4$ `

For fixed point, we have $x = x - x^2 - 4$, which implies $x^2 + 4 = 0 \Rightarrow x = \pm 2i$

$\Rightarrow$ the system has no fixed point.

Example: Consider the case of Logistic map

$x_{n+1} = \lambda x_n (1 - x_n)$

For fixed point of this system write $x = \lambda x(1 - x)$. Solving this, we obtain solutions 0, and $\frac{\lambda - 1}{\lambda}$. Thus, there are two fixed points of Logistic map system; $x_1^* = 0$ and $x_2^* = \frac{\lambda - 1}{\lambda}$.

Two dimensional discrete systems: $x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n)$

To find fixed point put

$x_{n+1} = x = f(x, y)$ and $y_{n+1} = y = g(x, y)$
Then, solve these resulting equations for pair of solutions (x, y). Only real solutions of these equations will be the termed as fixed points of the given system.

Example: Consider the Hénon map

$x_n+1 = 1 + y_n - a\, x_n^2$

$y_{n+1} = b\, x_n$

Proceeding with arguments similar to one dimensional case, for fixed points of this map, we substitute:

$x = 1 + y - a\, x^2$ & $y = a\, x$.

Then, after solving these, we obtain solutions (1, a) and $(-\frac{1}{a}, -1)$. Thus, we have two fixed points $p_1^* = (1, a)$ and $p_2^* = (-\frac{1}{a}, -1)$.

Three dimensional discrete systems: For three dimensional systems,

$x_{n+1} = f(x_n, y_n, z_n)$,

$y_{n+1} = g(x_n, y_n, z_n)$,

$z_{n+1} = h(x_n, y_n, z_n)$

we proceed similarly for fixed points and put

$x = x_{n+1} = f(x, y, z)$,

$y = y_{n+1} = g(x, y, z)$,

$z = z_{n+1} = h(x, y, z)$

Example: Consider the equations of a Food-Chain represented by:

$x_{n+1} = a\, x_n (1-x_n) - b\, x_n\, y_n$

$y_{n+1} = c\, x_n\, y_n - d\, y_n\, z_n$

$z_{n+1} = r\, y_n\, z_n$

Writing $x_{n+1} = x = ax(1-x) - b\,x\,y$, $y_{n+1} = y = c\,x\,y - d\,y\,z$ & $z_{n+1} = z = r\,y\,z$ and solving these, we get fixed points for the Food-Chain system shown above.

(b) Fixed points for continuous system

To find fixed points for continuous systems, put time derivatives equal to zero.

One dimensional systems: $\frac{dx}{dt} = f(x,t)$, put $\frac{dx}{dt} = 0$ and then solve the resulting equation for x.

Example: $\frac{dP}{dt} = kP(1-\frac{P}{M})$; put $\frac{dP}{dt} = 0$. Then fixed points are $P_1^* = 0$ and $P_2^* = M$.

Example: $\frac{dx}{dt} = x(x^2-1)$; put $\frac{dx}{dt} = 0$. Then fixed points are $x_1^* = 0$ and $x_2^* = 1$ and $x_3^* = -1$.

Two dimensional systems: $\frac{dx}{dt} = f(x,y,t), \frac{dy}{dt} = g(x,y,t)$, put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$ and then solve the resulting two equations for (x, y).

Example: $\frac{dx}{dt} = ax - bxy, \frac{dy}{dt} = -cy + dxy$ (Predator-Prey Model)

Put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, then the solutions of above Predator-Prey system be given by (0, 0) and $\left(\frac{c}{d}, \frac{a}{b}\right)$. Therefore, fixed points of this system are

$P_1^* = (0, 0)$ and $P_2^* = \left(\frac{c}{d}, \frac{a}{b}\right)$.

Three dimensional systems: $\frac{dx}{dt} = f(x,y,z,t), \frac{dy}{dt} = g(x,y,z,t), \frac{dz}{dt} = h(x,y,z,t)$

One needs to put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$,. $\frac{dz}{dt} = 0$ and then solve the resulting three equations for (x, y, z).

Example: The Lorenz system

The Lorenz system is a system of ordinary differential equations (the Lorenz equations) first studied by Edward Lorenz. It is notable for having chaotic solutions for certain parameter values and initial conditions. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system which, when plotted, resembles a butterfly as shown in figure 1.2.

$$\frac{dx}{dt} = \sigma(y-x)\,,\ \frac{dy}{dt} = rx - xz - y\,,\ \frac{dz}{dt} = xy - bz$$

This system evolves chaotically for σ = 10, r = 28, b = 8/3 and results in a chaotic attractor called Lorenz Attractor which has the shape of a butterfly. Due to this, it is known as a chaotic phenomenon, also termed as the “Butterfly Effect”.

Equilibrium points (fixed points):

Equilibrium points (fixed Points) of Lorenz equations are real solutions of the system of equations when time derivatives are put equal to zero. Then, one gets the equilibrium points denoted as $x_0 = (0, 0, 0)$, and with r > 1, the other two fixed points

$x_{c1} = (\sqrt{b(r-1)}\ ,\ \sqrt{b(r-1)}\ ,\ r - 1),$

$x_{c2} = (-\sqrt{b(r-1)}\ ,\ -\sqrt{b(r-1)}\ ,\ r - 1)$

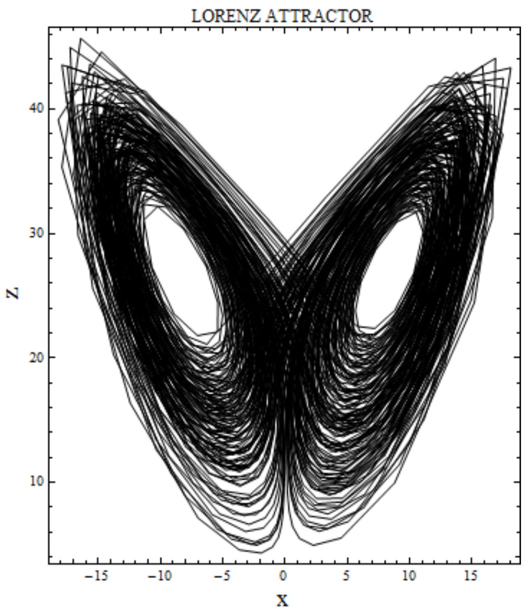


Fig.1.2. Lorenz attractor for σ = 10, r = 28, b = 8/3.

1.4.7 Stability & Asymptotic stability of Fixed Points

A fixed-point x^* is said to be stable if the orbit initiating nearby x^*has the tendency to converge to x^*.

Mathematically, a fixed-point x^* is stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\| x_0 - x^*\| \leq \delta$ implies that $\| x_n - x^*\| \leq \varepsilon$ for all $n \geq 1$.

In other words, once we have chosen how close we want to remain to x^* in the future (for a choice of ε), we can find how close we must start at the beginning (existence of δ).

The symbol, $\| . \|$ stands for "Euclidean norm" and that

$\| x \| = \|(x_1, x_2, x_3, . ., x_q . .) \| = (x_{12} + x_{22} + . . . + x\ q_2. .)$ ½

(a) Condition of stability for Discrete One-Dimensional Systems:

Let x^* be a fixed point of system $x_{n+1} = f(x_n)$ and x_0 be any initial point nearby x^*such that

$x_0 = x^* + \Delta$, where Δ is very small. Then, expanding

$f(x_0) = f(x^* + \Delta)$, about x^*, we get

$f(x_0) = f(x^* + \Delta) = f(x^*) + \Delta\ (df/dx)|x^*+\ldots$

Put $x_1= f(x_0)$ and omitting higher order small terms, as $x^* = f(x^*)$, we get

$(x_1 - x^*) = \Delta\ (df/dx)|x^*= (x_0 - x^*)\ (df/dx)|x^*$

$\Rightarrow$The distance to the fixed point decreased by the map if

$|(x_1 - x^*)| < |(x_0 - x^*)|$

i.e. If $|(x_1 - x^*)/(x_0 - x^*)| = |(df/dx)|x^* < 1$

or $|\ f'(x)|x^* < 1$, x^* is stable. Also, if $|\ f'(x)|x^* > 1$, x^* is unstable.

Example: Consider the case of Logistic map

$x_{n+1} = \lambda\ x_n\ (1 - x_n)$

For fixed point of this system write $f(x) = \lambda x(1 - x)$. Solving this, we obtain solutions 0, and $\frac{\lambda - 1}{\lambda}$. Thus, there are two fixed points of Logistic map system; $x_1^* = 0$ and $x_2^* = \frac{\lambda - 1}{\lambda}$.

Since $f'(x) = \lambda - 2\lambda x = \lambda(1 - 2x)$. Therefore,

at $x_1^* = 0$, $|f'(x)| = |\lambda|$. So if $|\lambda| < 1$, x_1^* is stable

$\Rightarrow$ As $\lambda > 0$, so if $0 < \lambda < 1$, x_1^* is stable

At $x_2^* = \frac{\lambda - 1}{\lambda}$, $|f'(x)| = |\lambda(1 - 2(\frac{\lambda - 1}{\lambda}))| = |2 - \lambda| = |\lambda - 2|$.

Therefore, if $|\lambda - 2| < 1$ then x_2^* is stable.

$\Rightarrow$ $1 < \lambda < 3$, then x_2^* is stable.

Example: Consider the map

$x_{n+1} = a - x_n^2$ or $f(x) = a - x^2$ (1)

For fixed point, we have $x = a - x^2$, which implies $x = \frac{-1}{2} \pm \frac{\sqrt{1+4a}}{2} \Rightarrow$ The system (1) has fixed points $x_1^* = \frac{-1}{2} + \frac{\sqrt{1+4a}}{2}$ and $x_2^* = \frac{-1}{2} - \frac{\sqrt{1+4a}}{2}$

From (1) we have, $f'(x) = -2x$.

At $x_1^* = \frac{-1}{2} + \frac{\sqrt{1+4a}}{2}$, we have $|f'(x)| = |-1 + \sqrt{1+4a}| = |\sqrt{1+4a} - 1|$

So, if $|\sqrt{1+4a} - 1| < 1$, $\Rightarrow$ $0 < \sqrt{1+4a} < 2 \Rightarrow 0 < 1 + 4a < 4$

$\Rightarrow -1 < 4a < 3 \Rightarrow \frac{-1}{4} < a < \frac{3}{4}$, then x_1^* is stable

At $x_2^* = \frac{-1}{2} - \frac{\sqrt{1+4a}}{2}$, we have $|f'(x)| = |-1 - \sqrt{1+4a}| = = |\sqrt{1+4a} + 1|$.

So, if $|\sqrt{1+4a} + 1| < 1$, $\Rightarrow -2 < \sqrt{1+4a} < 0 \Rightarrow 2 > -\sqrt{1+4a}$

$\Rightarrow 4 < 1 + 4a \Rightarrow a > \frac{3}{4}$, then x_2^* is stable.

(b) Condition of stability for Continuous One-Dimensional Systems:

For one dimensional continuous system $\dot{x} = f(x)$, The fixed point $\bar{x}$ is stable if $f'(\bar{x}) < 0$ and is unstable if $f'(\bar{x}) > 0$.

For two-dimensional system given as

$$f'(x) = f(x_1, x_2)$$
$$g'(x) = g(x_1, x_2)$$

where right hand side functions are smooth curves.

The Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}$$

Generally, we have to find the eigenvalues of this 2 x 2 matrix. If the largest real part of all the eigenvalues is positive then the fixed point is unstable and if the largest real part of all the eigenvalues is negative then the fixed point $(\bar{x}, \bar{y})$ is stable. If finding eigen values is a tedious task, then we can find stability of fixed points by using trace T and determinant D rule for the Jacobian matrix.

If the Jacobian matrix evaluated at the fixed point of interest is:

$J(\bar{x}, \bar{y}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $T = a + b$ and $D = (a.d - b.c)$

the fixed point can be one of six types:

If $D < 0$, the fixed point is a saddle point.

If $D > 0$ and If $T > 0$, the fixed point is unstable.

If $D > 0$ and If $T < 0$, fixed point is a stable.

If $D > 0$ and If $T = 0$, fixed point is called center.

If $D > 0$ and $T^2 - 4D > 0$ then the fixed point is a node.

If $D > 0$ and $T^2 - 4D < 0$ then the fixed point is a focus.

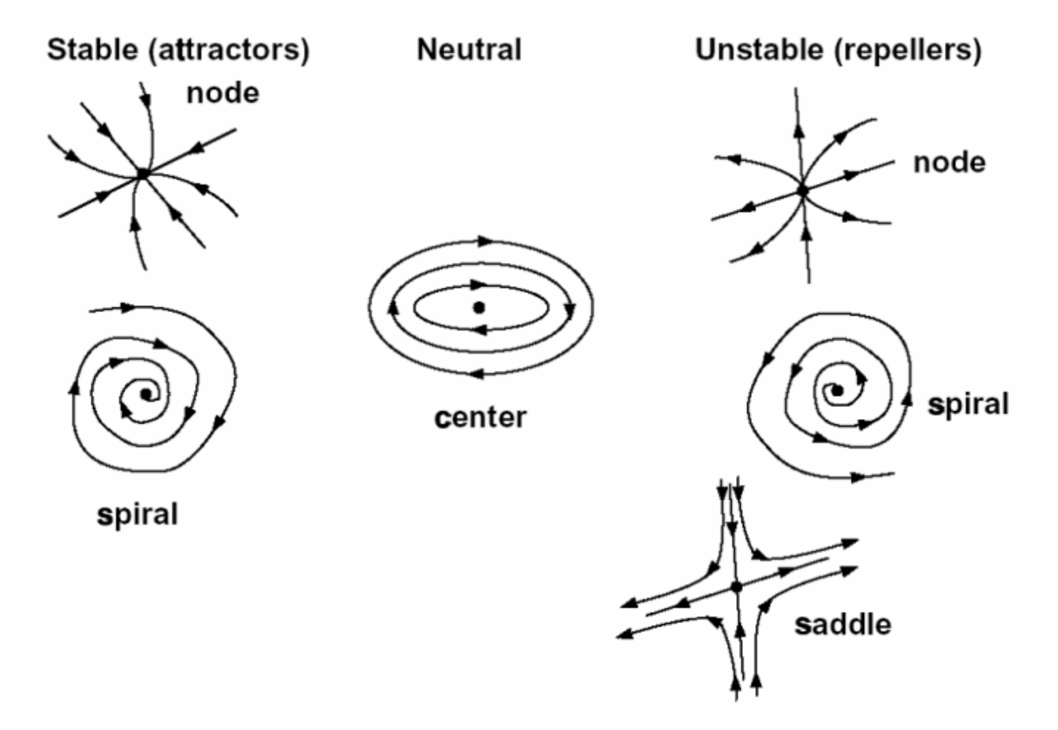


Fig. 1.3 Examples of equilibrium points.

For 3-dimensional system:

The Jacobian matrix of a three-dimensional system has 3 eigenvalues, one of which must be real and the other two can be either both real or complex-conjugate. From he signs and the types of the eigenvalues, let us decide the type of fixed points. The fixed point can be classified as:

Node when all eigenvalues are real and have the same sign; The node is stable (unstable) when the eigenvalues are negative (positive).

Saddle when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable.

Focus-Node when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive).

Saddle-Focus when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable.

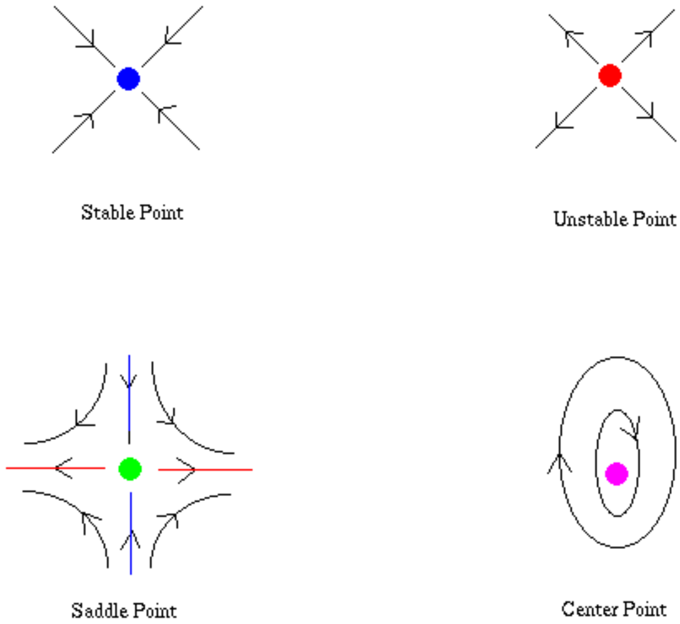


Fig. 1.4 Different types of fixed points.

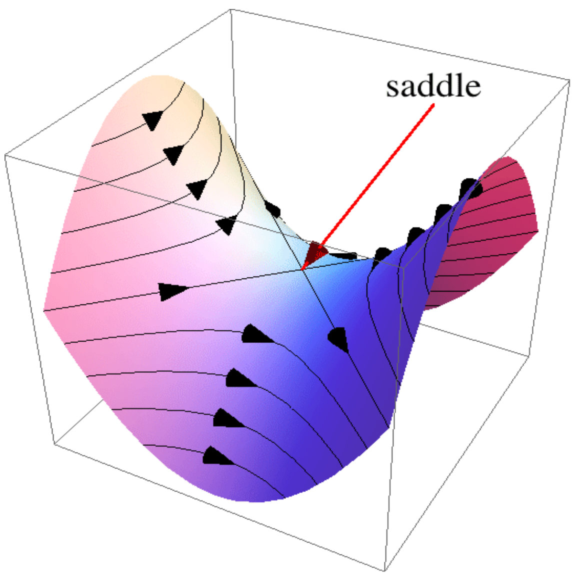


Fig. 1.5 Saddle point.

Stability of Orbit:

Let x^* be a fixed point of system f(x) and x_0 is an arbitrary point in the neighbourhood of x^*. Then, if the orbit $\{x_0, f_1(x_0), f_2(x_0), f_3(x_0), . . .\}$,

satisfies $| f_n (y) – x^* | = 0$

then x^* is said to be stable and the above orbit is known to be a stable orbit.

This also implies that two orbits of "f" with initial points x and y, both in the neighbourhood of x^*, approach asymptotically.

If all solutions of the dynamical system that start out near a fixed point x^* stay near x^* forever, then x^* is Lyapunov stable.

More strongly, if x^* is Lyapunov stable and all solutions that start out near x^* converge to x^*, then x^* is asymptotically stable.

1.4.8 Limit cycle

In analysis of non-linear systems, singular points are not the only interesting points that one may like to learn about. Many non-linear systems although unstable exhibit limit cycling. Even though trajectories stay bounded, they experience sustained and bounded oscillations. A limit cycle is a periodic orbit of a continuous dynamical system that is isolated for example the swings of a clock. Determining limit cycle is as important as finding fixed points in non-linear dynamics.

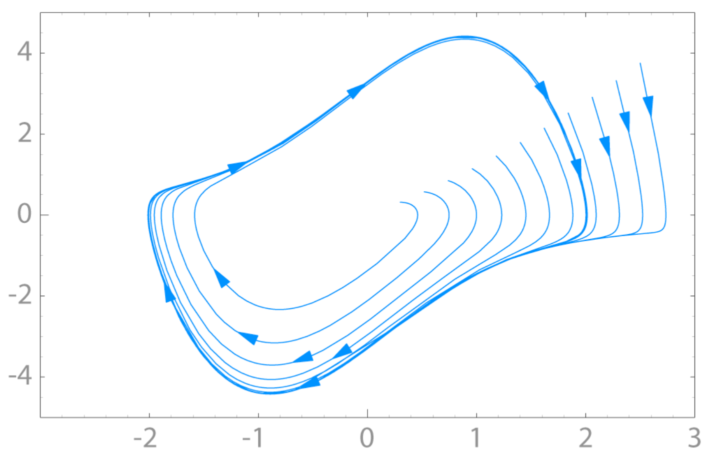


Fig.1.6 Stable limit cycle for Wander Pal Oscillator.

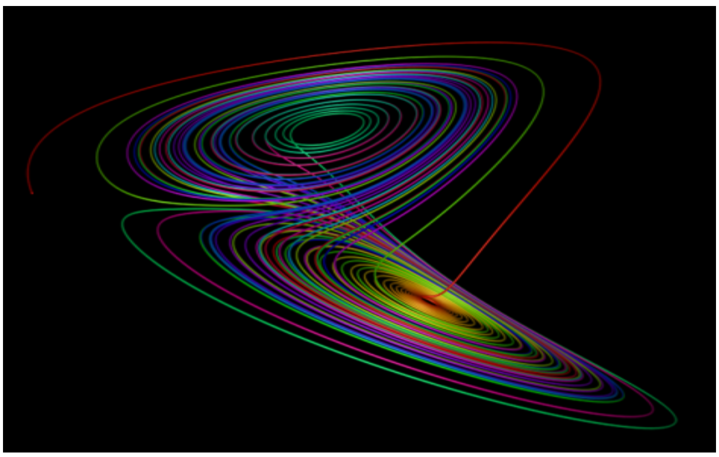
1.4.9 Attractors and Strange Attractor

Attractor: A Stable fixed point, a stable periodic orbit, a stable torus etc. are attractors of a dynamical system because these attract all nearby trajectories.

Strange Attractor: Strange attractor is a set and is also an attractor but with following special properties:

- An orbit originating within the set remains within it.
- Set is dense.
- Set has fractal self-similar property i.e. any small portion of the set, when magnified, provides same picture.

An attractor is called strange, if it's dimension isn't a natural number. Most (may be not all) strange attractors describe a chaotic movement. It has sensitive dependence on its initial conditions. If we plot the system's behaviour in a graph over an extended period we may discover patterns that are not obvious in the short term. In addition, even if we start with different initial conditions for the system, we will usually find the same pattern emerging. Strange attractors are unique since one does not know exactly where on the attractor the system will be. This term was coined by David Ruelle and Floris Takens (1971) A strange attractor has a fractal structure which means repeating itself while representing a complex pattern of behaviour in a chaotic system. These attractors are dense sets having fractal (fraction, non-integer) dimensions and satisfy dense property i.e. a self-similarity property.



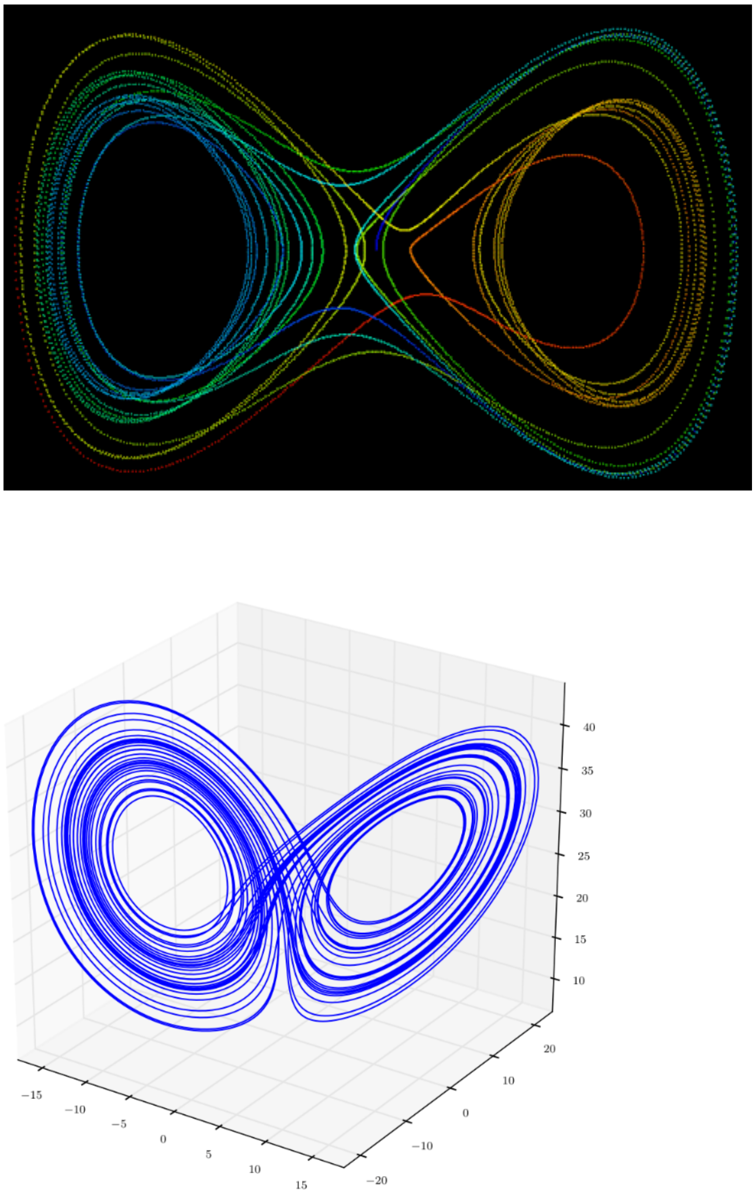


Fig. 1.7 Some Strange Attractors including Lorenz and duffing attractor.

1.4.10 Orbits and Periodic Orbits

Orbit: An orbit of a system $x_{n+1} = f(x_n)$ under rule f starting from point x_0 is the sequence of iterations $x_0, x_1, x_2, x_3, x_4, \ldots\ldots, x_k, x_{k+1}, \ldots$

Periodic Orbit:

An orbit $O(x_0)$ is said to be periodic of period $p \geq 1$ if $x_p = x_0$. The smallest integer p such that $x_p = x_0$ holds is called the minimum period of the orbit.

A periodic orbit of period p is said to be stable if each point x_i , i= 0, 1, 2, . . , p-1 is a stable stationary state of the dynamical system.

1.4.11 Food Web and Food chain

A food chain is a linear sequence of organisms through which nutrients and energy pass as one organism eats another. In a food chain, each organism occupies a different trophic level, defined by how many energy transfers separate it from the basic input of the chain. Food webs consist of many interconnected food chains and are more realistic representations of consumption relationships in ecosystems. Energy transfer between trophic levels is inefficient—with a typical efficiency around 10%.

This inefficiency limits the length of food chains.

1.4.12 Time Series

A time series is a series of points that are graphed in time order. Usually a time series is sequence taken at successive equally spaced points in time. Thus, it is a sequence of discrete-time data. A time series is a chart that plots the changing values of the variable(s) on the y-axis and time on the x-axis. Time series charts do play an important part in complexity science (J.C. Sprout 2003).

Epidemic Model $f(x) = k x^2 - 1$

Predator-Prey: $f(x,y) = ax(1-x) - b xy$, $g(x,y) = cy(1-y) + b xy$

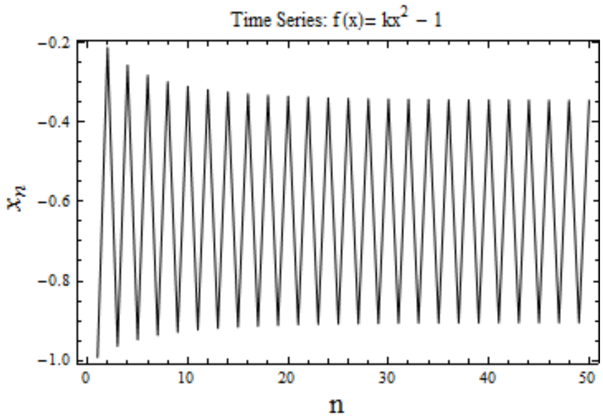


Fig.1.8(a): Time series plot for regular motion for $f(x) = k x^2 - 1$.

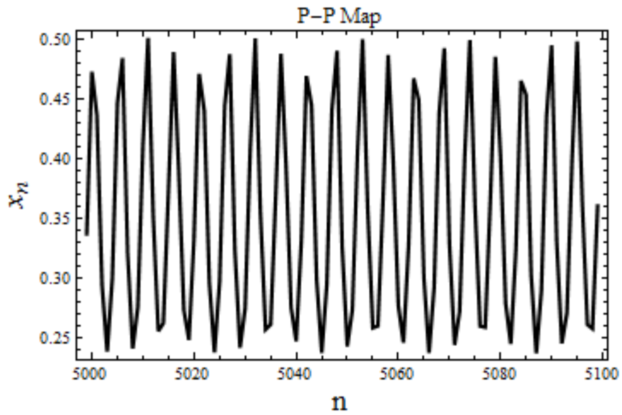


Fig.1.8(b): Time series plot for regular motion for predator prey model.

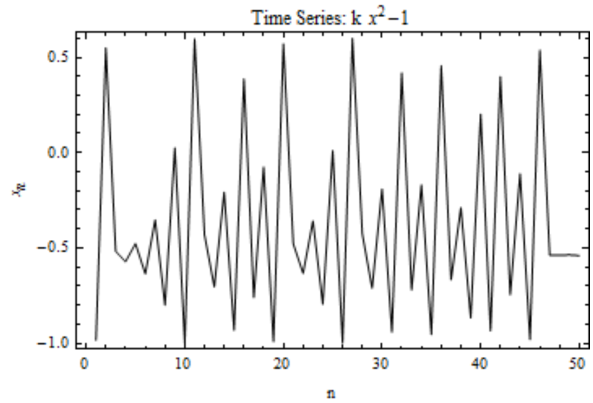


Fig.1.8(c): Time series plot for chaotic motion for f(x) = k x^2 – 1.

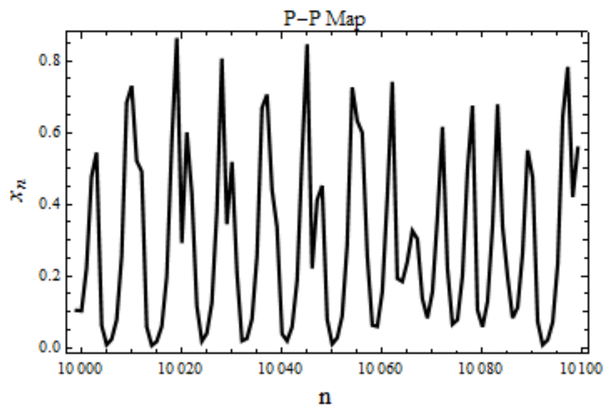


Fig.1.8(d): Time series plot for chaotic motion for predator prey model.

1.4.13 Phase Plots

Unlike time series plots, a phase space diagram plots the variables against each other and leaves time as an implicit dimension not explicitly graphed. A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.

Phase plots were originally stated by an American physicist J. Willard Gibbs. Phase portraits are an invaluable tool in studying dynamical systems. A phase plot of a dynamical system depicts the system's trajectories and stable steady states and unstable steady states in a state space. When the data is plotted in phase space with points in phase space representing the value of each of the variables at each moment of time, as the system changes over time, the data points make up a trajectory that is called a phase portrait.

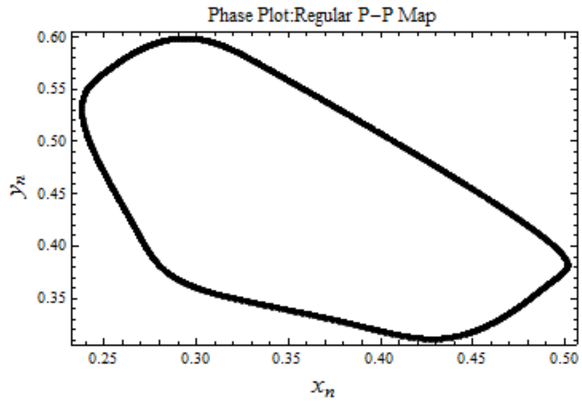


Fig. 1.9(a) Phase plots for regular attractor.

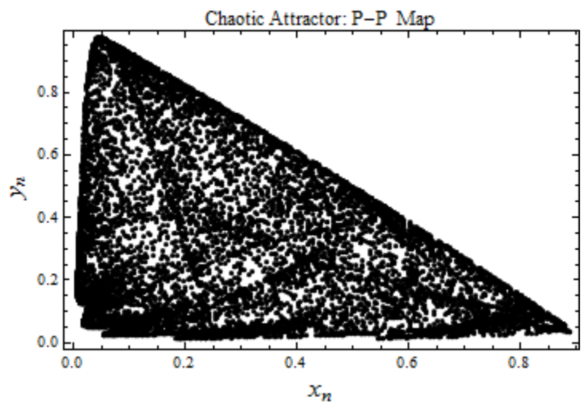


Fig. 1.9(b) Phase plots for chaotic attractor.

1.4.14 Continuous Map: Pendulum Equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \alpha x + \beta x^3 = \Gamma\cos\omega t$$

For k = 0.2, α = 0, β = 1, Γ = 10, ω = 1, we have chaos shown below

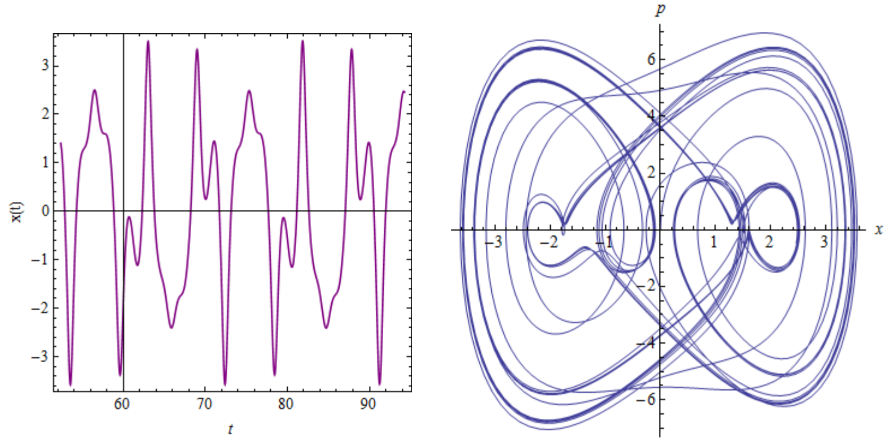


Fig.1.10: Equation of pendulum showing chaos.

1.5 Objectives of the present thesis

The objective of our research was to study the regularity and chaos in some biologically interactive models. Here, we proposed some mathematical models which are best fitted to the system and observed their evolutionary dynamics. Then we did numerical investigations by using "MATHEMATICA" software. For all the theoretical study we used the principles and concepts defined in nonlinear dynamics. Following are the main objectives of our research:

- To determine the fixed points of model according to the nature of parameters comprising the system.
- To study their stability and classification.
- To find suitable parameter values as well as initial conditions while iterating the function for regular evolution as well as for chaotic evolution.
- To find certain conditions on parameters as well as on initial values to change the behaviour of the system e.g. in case of controlling chaos.
- To make certain conclusions as well as predictions on the systems.

1.6 Organization of thesis

The first chapter gives a small introduction on nonlinear dynamics, chaos and regularity followed by importance of the area of research. Concepts like finding fixed points and determining their stability, classification of fixed points according to their stability criteria has been explained. Strange attractors which are chaotic sets with fractal behaviour have also been talked briefly. Further some research work has been reviewed followed by basic definitions, concepts and objectives of our research.

The second chapter sheds some light upon the research methodology. Many useful measures like bifurcation analysis, correlation dimension, topological entropy and LCEs which are very important for detailed studies of chaos and regularity have been explained.

In chapter 3, Complexity Measure in Simple Type Food Chain System has been investigated both analytically and numerically. Regular and chaotic motions have been observed for certain sets of values of a parameter of the system. For some further study, the continuous model of food chain has been transformed into discrete model by using Euler's method. Various measurable quantities for emergence of chaos, like Lyapunov exponents, topological entropies, correlation dimensions, have been numerically calculated and represented through plots. Finally, the chaos indicator, named Dynamic Lyapunov Indicator (DLI), has been used to identify clearly chaotic and regular motion.

In chapter 4, a problem on dynamics of two-gene Andrecut-Kauffman system has been studied for chaos and complexity. The model of this problem consists of nonlinear equations, in context of biochemical phenomena obtained from chemical reactions appearing in a two-gene model (Andrecut and Kauffman). Here, chemical reactions are assumed to correspond to gene expression and regulation. For this problem of two gene Andrecut-Kauffmann system, studies have been performed carefully to understand chaotic phenomena during its evolution together with complexities present in the system. Bifurcation analysis has been carried out to understand behaviour of steady state solutions leading to chaotic evolution. For chaotic and complex nature of evolution, numerical simulations have been performed to obtain Lyapunov exponents and topological entropies, for certain sets of parameters which explains the complexity and chaotic nature of evolution.

In chapter 5, a simple host-parasite type model has been considered to study the interaction of certain plants and herbivores. The two-dimensional discrete time model utilizes leaf and herbivore biomass as state variables. The parameter space consists of the growth rate of the host population and a parameter describing the damage inflicted by herbivores. Perceptive bifurcation diagrams, which give insightful results, have been presented here showing chaos and complexity in the system during evolution. Measure of complexity and chaos in the system is explained by performing numerical calculations and obtaining Lyapunov exponents, topological entropies and correlation dimension. Results are displayed through interesting graphics.

In chapter 6, we have worked upon a very interesting mature population model. In this chapter, we have worked with a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period. We have tried to analyse regularity and chaos in the system by finding bifurcations, topological entropy and Lyapunov exponents. The results and findings have been discussed at the end.

Chapter 7, is the last chapter where we have given a summary of our findings and the future prospect of this work.

Chapter 2

Methodology

2.1 Introduction

In this chapter we have talked about the methodology used for our work in detail Nonlinearity is present everywhere in physical phenomena. For this reason, an ever-increasing proportion of modern mathematical research is devoted to the analysis of nonlinear systems. They are vastly more difficult to analyse. In the nonlinear regime, many of the most basic questions remain unanswered for example the existence and uniqueness of solutions are not guaranteed. It is difficult to understand the explicit formulae. The principal of linear superposition does not hold. All the numerical approximations are not sufficiently accurate. But powerful computers have fomented a veritable revolution in our understanding of nonlinear mathematics.

To identify regular and chaotic evolution of a dynamical system, we have some tools like time series curves, phase plots, bifurcation diagrams, Lyapunov Exponents (LCE), topological entropies and correlation dimension. Since we have already discussed in our previous chapter about the time series curves and phase plots of dynamical systems, in this chapter we will talk in detail about bifurcations, LCE, topological entropy and correlation dimension which are very important to study chaos and regularity. In our work, we have majorly worked with these metrics to study chaos and its implications.

For a clear understanding of chaos, one has to observe the bifurcation diagrams obtained by varying certain parameters while keeping others fixed and drawing the results obtained along a particular variable. This helps a lot in understanding the flow of the system along with the bifurcation from one stable steady state to two, four, eight . . . leading to chaos. Once the system is chaotic, the next question is what is the extent of chaos strong, weak or very weak? Further, how do we measure this chaos? We have worked with a few efficient measures of regularity and chaos here for example Lyapunov characteristic exponents, topological entropy &

correlation dimension. We have also used the time series and phase plots while doing our work. Attractors have a big role to play in chaos theory as discussed in chapter 1.

Another very effective indicator of chaos which can be used is DLI which is now considered as an accurate indicator. Once we identify chaos in a system, we need to find out suitable methods to control the chaos and bring regularity in the system. Different techniques work in different systems. So, while doing all this analysis we have to be very sure of our numerical simulations. Hence apt mathematical methods and a strong conceptual framework is a must in this regard.

The Lyapunov exponents, (also known as Lyapunov Characteristic Exponents or LCEs), actually provide a measure of regular and chaotic motion. The positivity of LCE implies the motion is chaotic means two orbits originating nearby show divergence of behaviour. Predictability fails if LCE >0. But, negative value of LCE implies the motion is regular or periodic. In such a case one can predict the evolution.

2.2 Mathematical Calculation of LCE for a Discrete Map $x_{n+1} = f(x_n)$.

Recalling the calculations, we have done during our stability analysis of a fixed point, if x^* be a fixed point of system $x_{n+1} = f(x_n)$ and x_0 be any initial point nearby x^*such that $x_0 = x^* + \delta_0$, where δ_0 is very small distance of x_0 from x^*.

Denoting $x_1 = f(x_0) = f(x^* + \delta_0)$, and expanding we get

$x_1 = f(x_0) = f(x^* + \delta_0) = f(x^*) + \delta_0 (df/dx)|x^*+\ldots$

Omitting higher order small terms, as $x^* = f(x^*)$, we get

$$(x_1 - x^*) \approx \delta_0|f'(x_0)|$$

which means $\delta_1 = \delta_0|f'(x_0)|$

where $\delta_1 = (x_1 - x^*)$ = distance of x_1 from x^*.

If x_0 be a periodic point of period k, and two orbits start nearby at x_0 and at $x_0 \pm \delta_0$. Then after one iteration the distance between the two is approximated by

$$\delta_1 \approx |f'(x_0)|\delta_0 = M_0\delta_0$$

where M_0 is called the magnification factor for first step. At the second step

$$\delta_2 \approx |f'(x_1)|\delta_1 = M_1\delta_{1=}M_1M_0\delta_0$$

where M_1 is the magnification factor for the second step. Continuing in this manner, we conclude that the total magnification factor over one cycle of the period k orbit is the product

$$M_0 \, M_1 \, . \, . \, . \, M_{k-1}$$

Since this product is an accumulation of magnification factors, it makes sense to consider some average of it. The most convenient is the geometric average

$$(M_0M_{1\dots\dots\dots}M_{k-1})^{\frac{1}{K}}$$

which by taking logarithms leads to the arithmetic average.

$$\lambda = log(M_0M_{1\dots\dots\dots}M_{k-1})^{\frac{1}{K}}$$

$$\lambda = \frac{1}{k}(logM_0 + logM_1 + \cdots \dots . logM_{k-1})$$

$$= \frac{1}{k}(\log|f'(x_0)| + \log|f'(x_1)| + \cdots \log|f'(x_{k-1})|)$$

Condition for stability of a periodic orbit is that the average magnification factor is less than 1, which is equivalent to $\lambda < 0$ (stable) & $\lambda > 0$ (unstable).

Definition: Lyapunov exponent

Let f be a smooth map on R and let x_0 be a given initial point. Lyapunov exponent λ (x_0) of a map f is given by

$$\lambda(x_0) = \lim_{k\to\infty} \frac{1}{k} (\log|f'(x_0)| + \log|f'(x_1)| + \cdots \log|f'(x_{k-1})|)$$

Example: Consider the logistic map f(x) = λ x (1 – x), for 1 ≤ λ ≤ 4, the plot of Lyapunov exponents is given below.

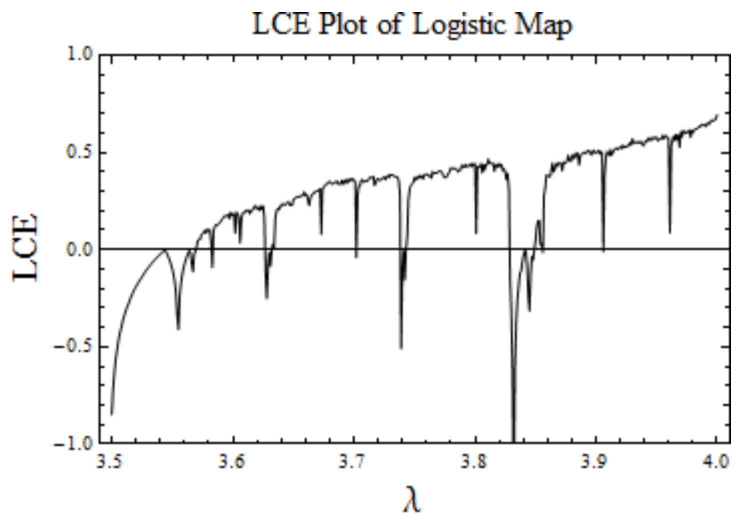


Fig.2.1 LCE plot of Logistic Map.

2.3 Bifurcation analysis

In ordinary sense, bifurcation means splitting into two. Similar behaviour also occurs in dynamical systems and so is the name. In case of a dynamical system, bifurcation occurs when a small smooth change is made to values of certain parameters of the system. A stable fixed point becomes unstable at some step and suddenly two stable solutions appear i.e. one stable cycle splits into two stable cycles, then 2-cycles becomes 4-cycles and so on. Actually, one observes the phenomena of sudden 'qualitative' or topological change in the behaviour of the system.

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a system during evolution. A bifurcation of a dynamical system is a phenomenon to observe qualitative change in its dynamics produced by varying parameters. This theory provides a strategy for investigating the bifurcations that occur within a family.

Bifurcation literally means splitting into two. The name "bifurcation" was first introduced by Henri Poincaré in 1885. In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden 'qualitative' or topological change in its behavior. Bifurcations occur in both continuous systems and discrete systems. Bifurcation(s) result when certain parameters on the dynamical equations, that is conditions affecting the system, reach critical thresholds (May, 1976). Usually, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets change. Bifurcations are of two types:

i) Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters crossing through critical thresholds.

ii) Global bifurcations, which often occur when larger invariant sets of the system collide with each other, or with equilibria of the system.

The nature of the solutions to a dynamical system, defined by a suitable differential equation, may change abruptly as a function of some control parameter. The most commonly observed transitions in dynamical states are “bifurcations”. A careful study of bifurcation diagram can lead to many observations in a system (Khalil, 2002, Kuznetsov et al., 2003, S., 2013).

Starting with initial conditions, with change in time, evolution in a dynamical system takes place for a certain set of parameter values. Now if we vary a single parameter while keeping other parameters fixed and then observe the change in the system along one of its particular variable and then draw the corresponding graphics and by observing this we can analyse the behaviour of the system. In this process we find a stable fixed point which becomes unstable for a while and then suddenly changes into two stable fixed points at some parameter value. This is a qualitative change at a particular value of the parameter. Also, at these parameter values one cycle splits into two cycles and hence we see a bifurcation. This point is called a bifurcation point. Continuing the processes, we find another parameter values where two cycles split into four cycles, then eight, sixteen, thirty-two etc. After a while, the system becomes regular because of the property of predictability. When there is no more predictability, the system is unpredictable and chaotic. The bifurcation diagram illustrates all the information of this evolution. Thus, in the system one observes period doubling which leads to chaos.

We have used the following code to find bifurcations for the Plant-Herbivore model where we worked with MATHEMATICA.

```
$MaxExtraPrecision = 1000000;

TransientPoint[fTP_, initcondTP_, μTP_, transTP_] := Nest[fTP[#, μTP] &, initcondTP, Round[transTP]];

PointsList[fPL_, initcondPL_, μPL_, npointsPL_] := NestList[fPL[#, μPL] &, initcondPL, Round[npointsP

ShowAttractor[fSA_, initcondSA_, μSA_, transSA_, npointsSA_,
   optionsSA_: {{1, 2}, AxesLabel -> {"First Variable", "Second Variable"}}] :=
  Module[{iSA, jSA, x0SA, tableSA, plotSA}, x0SA = TransientPoint[fSA, initcondSA, μSA, transSA];
   tableSA = PointsList[fSA, x0SA, μSA, npointsSA];
   plotSA = Table[ToExpression[FromCharacterCode[{iSA, 65, 108, 105}]], {iSA, 97, 96 + Length[optionsS
   For[jSA = 1, jSA <= Length[optionsSA],
    plotSA[[jSA]] =
     ListPlot[Table[{tableSA[[iSA, optionsSA[[jSA, 1, 1]]]], tableSA[[iSA, optionsSA[[jSA, 1, 2]]]]}
      {iSA, 1, Length[tableSA]}], Rest[optionsSA[[jSA]]]]; jSA++]; plotSA];

BifurcationTable[fBT_, initcondBT_, transBT_, npointsBT_, {paraMinBT_, paraMaxBT_, nparaBT_: 1
  Module[{iBT, pBT, dpBT, x0BT, tableBT}, tableBT = {}; dpBT = (paraMaxBT - paraMinBT)/(Round[nparaBT]); x0BT = init
   For[iBT = 1, iBT <= Round[nparaBT], pBT = paraMinBT + iBT dpBT;
    x0BT = TransientPoint[fBT, x0BT, pBT, transBT];
    tableBT = Append[tableBT, {pBT, PointsList[fBT, x0BT, pBT, npointsBT]}];
    x0BT = Last[Last[Last[tableBT]]]; iBT++]; tableBT];
```

```
ShowBifurcationDiagram[fSBD_, initcondSBD_, transSBD_, npointsSBD_,
   {paraMinSBD_, paraMaxSBD_, nparaSBD_: 1000},
   optionsSBD_: {1, AxesLabel -> {"Parameter Axis", "Variable Axis"}}] :=
  Module[{iSBD, jSBD, kSBD, tableSBD, t1SBD, plotSBD}, t1SBD = {};
   tableSBD = BifurcationTable[fSBD, initcondSBD, transSBD, npointsSBD,
     {paraMinSBD, paraMaxSBD, nparaSBD}];
   plotSBD = Table[ToExpression[FromCharacterCode[{iSBD, 65, 108, 105}]],
     {iSBD, 97, 96 + Length[optionsSBD]}];
   For[jSBD = 1, jSBD <= Length[optionsSBD],
    For[kSBD = 1, kSBD <= Length[tableSBD[[1, 2]]],
     t1SBD = Join[t1SBD, Table[{tableSBD[[iSBD, 1]], tableSBD[[iSBD, 2, kSBD, optionsSBD[[jSBD, 1]]]]}
        {iSBD, 1, Length[tableSBD]}]]; kSBD++]; plotSBD[[jSBD]] = ListPlot[t1SBD, Rest[optionsSBD[[jS
    t1SBD = {}; jSBD++]; plotSBD];
```

Deterministic Model : 1

$$x_{n+1} = x_n e^{r(1-x_n)-a y_n}$$

$$y_{n+1} = x_n e^{r(1-x_n)}[1 - e^{-a y_n}]$$

Stochastic Model : 2

$$x_{n+1} = x_n e^{r(1-x_n)-a y_n} + k\xi$$

$$y_{n+1} = x_n e^{r(1-x_n)}[1 - e^{-a y_n}]$$

where ξ is a random number

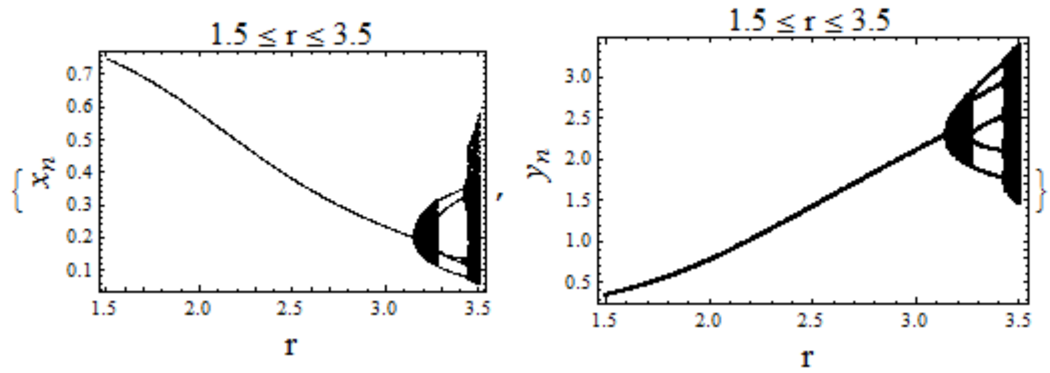
```
Clear[f, g, r, a, k, x, y]

f[x_, y_] = x Exp[r (1 - x) - a y];
g[x_, y_] = x Exp[r (1 - x)] * (1 - Exp[-a y]);

a = 1.0;

Plantheb[var_, rpara_] := {var[[1]] Exp[rpara (1 - var[[1]]) - a var[[2]]],
   var[[1]] Exp[rpara (1 - var[[1]])] * (1 - Exp[-a var[[2]]])};

Figure1 = ShowBifurcationDiagram[Plantheb, {0.2, 0.5}, 2000, 400, {1.5, 3.5, 500},
  {{1, Frame -> True, FrameLabel -> {StyleForm[" r ", FontSize -> 12], StyleForm["xn ", FontSize ->
    PlotRange -> All, Axes -> False, PlotStyle → {PointSize[0.005], Black}, PlotLabel → " 1.5 ≤ r ≤ 3
   {2, Frame -> True, FrameLabel -> {StyleForm[" r ", FontSize -> 12], StyleForm[" yn", FontSize ->
    PlotRange -> All, Axes -> False, PlotStyle → {PointSize[0.005], Black}, PlotLabel → " 1.5 ≤ r ≤ 3
```



In similar way, we have used MATHEMATICA codes for calculating Bifurcation diagrams for all the models and, also, to calculate Lyapunov exponents, Topological Entropies and Correlation dimensions.

2.4 Measure of chaos

Introducing a quantitative measure of chaos is important for several reasons. Most importantly it allows us to define exactly what we mean by chaos. We use measures like Lyapunov characteristic exponent, correlation dimension and topological entropy to do the same. Having such measures of chaos allows us to go further quantitatively and compare different systems. We can rationalize what we mean by saying that one system is more chaotic than another system. Thus, we can compare the extent of chaos of a system with different parameter values, or the chaos in two completely different systems.

2.4.1 Lyapunov Characteristic Exponent

Lyapunov exponents is a quantitative measure of the exponential divergence of nearby trajectories. When a Lyapunov exponents is positive, system is chaotic i.e. the trajectories diverge while a negative Lyapunov characteristic exponent indicates convergence of the trajectories (Strogatz, 1994). The systems show a strange attractor for certain parameter values. We calculate the dimensions of these attractors and see that the dimensions don't have to be an integer.

Consider any one-dimensional map defined in some interval (a, b)

$$x_{n+1} = f(x_n)$$

and two of its orbits starting at x_0 and $x_0 + \delta_0$ where δ_0 is very small. Then, expanding $f(x_0+\delta_0)$ by Taylor series, after one iteration we find the distance between the orbits as

$$\delta_1 = |f'(x_0)|\delta_0 = M_0\delta_0$$

M_0 is known as first step magnification factor. Similarly, at the second iteration, the distance between the orbits can be written as

$$\delta_2 = |f'(x_1)|\delta_1 = M_1\delta_1 = M_1M_0\delta_0$$

We continue for n iterations and separation between the orbits at n^{th} iteration is

$$\delta_n = |f'(x_{n-1})|\delta_{n-1} = M_{n-1}\delta_{n-1} = M_{n-1}M_{n-2} \dots M_0\delta_0$$

The product $M_{n-1}M_{n-2} \dots M_0\delta_0$
is the accumulation of magnification factors, so we consider its average value. The most convenient average to consider here is the geometric average

$$(M_{n-1}M_{n-2} \dots M_0\delta_0)^{\frac{1}{n}}$$

Taking log, one obtains the arithmetic average

$$\boldsymbol{\lambda = \ln(M_{n-1}M_{n-2} \dots \dots . M_0\delta_0)^{\frac{1}{n}}}$$

$$= \frac{1}{n}(\ln M_{n-1} + \ln M_{n-2} + \cdots)$$

$$= \frac{1}{n}(\ln|f'(x_0)| + \ln|f'(x_1)| + \ln|f'(x_2)| + \cdots + \ln|f'(x_{n-1})|)$$

If $\lambda < 1$, the orbit is stable and if If $\lambda > 1$ the orbit is unstable. For more accurate result, large iterations should be taken into account. This leads to the following definition of Lyapunov exponents.

Lyapunov exponents of a smooth map f on R with x_0 as initial point may be defined as:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}(\ln|f'(x_0)| + \ln|f'(x_1)| + \ln|f'(x_2)| + \cdots + \ln|f'(x_{n-1})|)$$

provided the limit exists.
Lyapunov number is the exponent of Lyapunov exponent and is given by

$$L(x_0) = e^{\lambda(x_0)}$$

From above definition, a clear interpretation for Lyapunov exponent is given as:

It is the measure of loss of information during the process of iterations.

For higher dimensional system, we can generalize the above one-dimensional case to obtain

$$\lambda(X_0, U_0) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{t=0}^{n-1} J(X_t) U_0 \right\|$$

and
$$\| X_n - Y_n \| \approx e^{\lambda(X_0, U_0) n},$$

where $X \in R^n$, $F: R^n \rightarrow R^n$, $U_0 = X_0 - Y_0$.
J is the Jacobian of the map F.
Quantitatively, two trajectories in phase space with initial separation δx_0 diverge if

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)|$$

where λ is the Lyapunov exponent. The system described by the map F is regular for $\lambda \le 0$ and chaotic when $\lambda > 0$.

For all numerical simulations to find all the LCEs during our calculations we have used MATHEMATICA and developed codes like this.

```
Clear[r, a, x, y]
f[x_, y_] := x Exp[r (1 - x) - a y];
g[x_, y_] := x Exp[r (1 - x)]*(1 - Exp[-a y]);
M[x_, y_] := {{E^(r (1 - x) - a y) - E^(r (1 - x) - a y) r x, -a E^(
r (1 - x) - a y) x}, {E^(r (1 - x)) (1 - E^(-a y)) -
E^(r (1 - x)) (1 - E^(-a y)) r x, a E^(r (1 - x) - a y) x}}
a = 1.5; r = 3.25
#q = #[#[1, 0},# [0, 1}};
x = 0.2; y = 0.5; n = 1;
```

```
q1 = Table[{n, u = x, n, v = y, q = M[x, y].q, n,
Log[(Max[Abs[Eigenvalues[q]]])^(1/n)], n = n + 1, x = f[u, v],
y = g[u, v]}, {10000}];
p = MatrixForm[q1[[Range[5000, 5030], Range[6, 7]]]]
k = q1[[Range[5000, 7000], Range[6, 7]]];
plot1 = ListPlot[k, PlotRange -> All,
PlotStyle -> {Thickness[0.008], Blue}, Frame -> True,
FrameLabel -> {"n", "LCE"}, Joined -> True, Ticks -> Automatic]
```

2.4.2 Topological Entropy

Lyapunov exponents are a great help in measuring chaos but they have a few limitations to their use. Lyapunov exponents are local in nature and are not necessarily constant throughout the evolution and so ergodicity is also required to characterize chaos. Conceptually, Lyapunov exponents are time dependent and so it isn't great for systems arising from relativistic considerations.
A chaotic attractor is composed of a complex pattern. To investigate chaotic behaviour in a wide variety of systems evolving with time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as an indicator is the topological entropy (Iwai, 1998, Balmforth et al., 1994, Gora and Boyarsky, 1991, Bowen, 1970). The concept of topological entropy was first introduced by Adler, Konhelm and McAndrew in the 1960s Topological entropy describes the rate of mixing of a dynamical system. It is related to Lyapunov exponents both through the dependence of rate and to the ergodicity.

For a system having non-zero topological entropy, the rate of mixing must be exponential which is comparable to Lyapunov exponents. Though such exponentiality is not relative to time, rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. The book by Nagashima and Baba (Nagashima and Baba, 1998) gives a very clear definition of entropy.

Consider a finite partition of a state space X denoted by

$A = \{A_1, A_2, \ldots\ldots\ldots\ldots.. A_N\}$.

Then a measure μ on X with total measure μ(X) = 1 defines the probability as $\boldsymbol{p_i = \mu(A_i)}$, i = 1,2,……..N.

$\mathbf{H(A) = -\sum_{i=0}^{N} p_i \, Log \, p_i}$

Then the entropy of the partition be given by

$$\mathbf{H(A) = -\sum_{i=0}^{N} p_i \, Log \, p_i}$$

The code for topological entropy used is described as follows:

```
Clear[r, a, x, n, y]
f[x_, y_] := x Exp[r (1 - x) - a y];
g[x_, y_] := x Exp[r (1 - x)]*(1 - Exp[-a y]);
a = 1.5; r = 2.25; x = 0.3; y = 0.9; n = 0;
entropy[p_List] := -p.Log[p]
Clear[r, a, x, n, y]
a = 1.5;
fxdata[r_] := (
x = 0.3; y = 0.9; xLL = {};
Do[t = i; { x = x Exp[r (1 - x) - a y];
y = x Exp[r (1 - x)]*(1 - Exp[-a y]) };
If[t < 400, Continue[], xLL = Append[xLL, x]], {i, 500}];
fxdata[r] = xLL)
prob[r_] :=
Select[1/500 BinCounts[fxdata[r], {0, 1, 1/500}], Positive]
entropylist = Table[{r, entropy[prob[r]]}, {r, 0.0, 0.8, 0.01}];
p2 = ListPlot[entropylist, PlotStyle -> {Thickness[0.01], Red},
PlotRange -> All, Joined -> True, Frame -> True,
FrameLabel -> {"r ", "\!\(\*
StyleBox[\"entropy\",\nFontSize->16,\nFontColor->GrayLevel[0]]\)\!\(\*
```

```
StyleBox[\" \",\nFontSize->16,\nFontColor->GrayLevel[0]]\)"},
Ticks -> Automatic, GridLines -> Automatic]
```

2.4.3 Correlation Dimension

In chaos theory, the correlation dimension is a measure of the dimensionality of the space occupied by a set of random points. Correlation dimension provides a measure of dimensionality of the chaotic attractor. It is a very practical and efficient method as compared to many other methods. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension also gives a measure of complexity for the underlying attractor of the system.

The calculation of correlation dimension involves some statistical concepts. Correlation dimension is calculated after drawing the correlation curves for each system. For this, first, we have collected data for the correlation curves plotted for each model and then used the method of least square linear fit described by Martelli (Martelli, 2011) and Nagashima and Baba (Nagashima and Baba, 1998) to obtain correlation dimensions.

Correlation dimension is a kind of fractal dimension and its numerical value is always non-integer. To calculate this, we have used the following procedure (Martelli, 2011).

Consider an orbit

$O(x_1)=\{x_1, x_2, x_3, x_4,\ldots\ldots\ldots\ldots\ldots\}$

for a map f: $U \rightarrow U$, where U is an open bounded set in R_n. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral,

$$\mathbf{C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H(r - \|x_i - x_j\|)}$$

where $\quad H(x) = \begin{cases} 0, x < 0 \\ 1, x \geq 0 \end{cases}$

is the Heaviside unit-step function. The summation indicates the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$.

Here, C (r) measures the density of pair of distinct vectors x_i and x_j that are closer than r. Finally, the correlation dimension D_c is defined as

$$\boldsymbol{D_c = lim_{r \to 0} = \frac{\ln C(r)}{\ln r}}$$

In practice, we select a large number of n states of orbit $O(x_1)$, and approximate $C(r)$ for several different values of r. Then the ratio $\frac{\ln C(r)}{\ln r}$ is plotted with r and the y-intercept of the line obtained by least square linear fit to the correlation integral data is regarded as a reasonable estimate of D_C where the slope is sufficiently small.

2.5 Chaos and regularity Indicators

While working with chaos theory and dynamical systems, even though the conventional tools like time series, phase plots, bifurcation diagrams etc. are the most commonly used still other indicators are also widely used as powerful tools in developing the theory and results. We have already talked about a few of these in this thesis.

Some more chaos indicators like Fast Lyapunov Indicator (FLI) and Smaller Alignment Indices (SALI) are also very useful during study of chaos and regularity. Dynamic Lyapunov Indicator (DLI) which is a new indicator has also been mentioned. A brief introduction of these indicators is as follows:

2.5.1 Fast Lyapunov Indicator

Fast Lyapunov indicator is a method which is based on the variation of the length of vectors evolving in tangential space with time, which distinguishes very quickly between regular and chaotic motion. FLI's increase exponentially for chaotic orbits and linearly for regular orbits.

Starting with m-dimensional basis,

$$V_m(0) = (v_1(0), v_2(0), \ldots\ldots\ldots v_m(0))$$

Embedded in n-dimensional space with an initial condition

$$x_1(0), x_2(0) \ldots\ldots\ldots\ldots x_m(0)$$

for each iteration we take the largest amongst the vectors of the evolving basis. Thus, the FLI is defined as:

$$\text{FLI} = \sup \|v_j\|, j = 1,2,\ldots,m$$

2.5.2 Dynamic Lyapunov Indicator

The Dynamic Lyapunov Indicator (DLI) is defined as the largest value estimated among all eigenvalues λ_j of the Jacobian matrix J such that $|J - \lambda_i I| = 0; j = 1,2 \ldots n$. (for n-dimensional map) of the examined map for all discrete times. We plot the largest eigen value at every time step of the evolving Jacobian matrix and we observe these eigenvalues.

If the eigenvalues form a definite pattern, then the motion is regular and if they are distributed randomly (with no pattern), then the motion is chaotic.

2.5.3 Smaller Alignment Index

The Smaller Alignment Index (SALI) is a very useful and efficient indicator that can distinguish rapidly and with certainty between ordered and chaotic motion. SALI behaves different in the two cases (chaos and regularity), as it fluctuates around non-zero values for ordered orbits and converges exponentially to zero for chaotic orbits.

First consider n-dimensional phase space and an orbit in this space with initial condition

$P(0) = (x_1(0), x_2(0), \ldots\ldots\ldots\ldots\ldots x_n(0))$

and a deviation vector

$\xi(0) = (dx_1(0), dx_2(0), \ldots\ldots\ldots\ldots\ldots dx_n(0))$

for the initial point $P(0)$.To compute the SALI for a given orbit, we follow the time evolution of the orbit of $P(0)$ together with two deviation vectors $\xi_1(t)$, $\xi_2(t)$ which initially points in two different directions in the phase space. At every time step, the two deviation vectors $\xi_1(t)$, $\xi_2(t)$ are normalized and the SALI is then defined as follows:

SALI = $$\min\left\{\left\|\frac{\xi_1(t)}{\|\xi_1(t)\|}-\frac{\xi_2(t)}{\|\xi_2(t)\|}\right\|,\left\|\frac{\xi_1(t)}{\|\xi_1(t)\|}+\frac{\xi_2(t)}{\|\xi_2(t)\|}\right\|\right\}$$

The SALI fluctuates around a non-zero value for ordered orbits while it tends to zero for chaotic orbit.

The measures mentioned above are useful indicators of regularity and chaos in a system. In our work, we have used these metrics to analyse the systems under study.

Chapter 3

Complexity Measure in Simple Type Food Chain System

3.1 Introduction

Study of ecological systems of nature are of growing importance in many areas of research these days. A real natural system is full of nonlinearity and its dynamics are very complex. Prey-Predator interactions have many applications of mathematics to biology. In this regard, Lotka-Volterra predator-prey model is a very simple model proposed by Lotka (Lotka, 1925) and Volterra (Volterra, 1926). Researchers have found very interesting evolutionary characters and suggested more studies in food chain systems. Three species food chain system was described by many researchers (Rosenzweig and MacArthur, 1963). Because of availability of effective computational techniques these days, much deeper investigations are possible and new interesting results may emerge. With time, techniques have been developed to obtain conditions for a model of a predator-prey system with mutual interference to possess a globally stable positive equilibrium (Freedman and So, 1985).

The appearance of chaos in a continuous time model of a food chain incorporating nonlinear functional model suggests that chaotic dynamics may be common in natural food webs (Hastings and Powell, 1991). Many researchers have worked with food chain models with varied ideas (Muratori and Rinaldi, 1992), (Boer et al., 1999), (De Feo and Rinaldi, 1998), (Deng and Hines, 2002). Studying ecological chaos is not a trivial task but mathematical models of food chains have made the study of theoretical chaos possible. Since there can be many constraints in terms of laboratory infrastructure or the costs associated with other techniques, studying the problems associated with arising and disappearing of chaos from food chains is a very attractive mathematical field. Such detailed and deep understanding can help in developing insights into evolutionary dynamics of predator prey models.

A detailed review of food chain research articles can be obtained from the article by Deng (Deng, 2001) and by Elsadany (Elsadany et al., 2012).

Study of complexities arising during evolution of a food chain system has been investigated here. We have worked with a classical mathematical model for three-trophic food chains modified and used by Bo Deng (Deng, 2006). The Rosenzweig-MacArthur food chain model (Rosenzweig and MacArthur, 1963) has been used by many authors for food chain chaos. Bo Deng modified Rosenzweig-MacArthur food chain model with the inclusion of intraspecific competing predators tending to stabilize food chains.

We have intended to study the complexities arising during evolution in this food chain system. Regular and chaotic motions have been observed for certain sets of values of a parameter of the system. For detailed further study, the continuous model of food chain has been transformed into discrete model by using Euler's method. Various measurable quantities for emergence of chaos, like Lyapunov exponents, topological entropies, correlation dimensions, have been numerically calculated and represented through plots. Finally, the chaos indicator, named Dynamic Lyapunov Indicator (DLI), has been used to clearly identify chaotic and regular motion.

3.2 The Model

Consider the following model for three-trophic food chain:

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - \frac{p_1 X}{H_1 + X}Y \qquad (1)$$

$$\frac{dY}{dt} = Y\left(\frac{b_1 p_1 X}{H_1 + X} - d_1 - s_1 Y\right) - \frac{p_1 Y}{H_2 + Y}Z \qquad (2)$$

$$\frac{dZ}{dt} = Z\left(\frac{b_2 p_2 Y}{H_2 + Y} - d_2 - s_2 Z\right) \qquad (3)$$

where, X, Y and Z denote the population densities, respectively for prey, predator, and top-predator populations. K is the carrying capacity of the prey population in absence of the predator population and in such a case r denotes the maximum per capita growth of the prey population.

The parameters b_1, b_2 are the birth- to –consumption ratios for predators; H_1, H_2 are the semi-saturation constants; p_1, p_2 are the maximum per capita capture rates; d_1, d_2 are per capita minimum death rate for each predator and the products s_1Y, s_2Z are additional per capita density dependent death rate of each predator. For a simple mathematical analysis and equivalent dynamics, we have used the dimensionless form.

Dimensionless approach typically generalizes the problem as dimensionless solution depends on a set of dimensionless parameters. Non-dimensionalising helps one to decide which are the relevant variables and how they might be related. The use of a dimensionless model is a common way to study a wide variety of physical phenomena or engineering problem, or even economical tasks. A dimensionless equation, (algebraic or differential), involves variables without physical dimension. The purpose of dimensionless equations is:

- To simplify the equation(s) by reducing the number of variables used.
- To analyse system behaviours regardless of the unit used to measure variables.
- To rescale parameters and variables so that all computed quantities are of same order (relatively similar magnitudes).

A dimensionless number is associated with a value that could be iterated by dimensional analysis.

However, the dimensionless equations have a great impact on nonlinear phenomena. In the linear case physical parameters are involved in transformations hence solving any single dimensional system can give an idea about all others having the same behaviour. This can be said for nonlinear systems only when they have same behaviour in non-dimensional form.

Dimensionless form often shows very clearly that not the original variables but rather their ratios (or other combinations) govern the qualitative type of solution.

We have used the following changes of variables and parameters to transform the equations to a dimensionless form:

$$t \to b_1 p_1 T$$

$$x = \frac{X}{K}, y = \frac{Y}{Y_0}, z = \frac{Z}{Z_0}, \beta_1 = \frac{H_1}{K}, \beta_2 = \frac{H_2}{K},$$

$$Y_0 = \frac{rK}{p_1},$$

$$Z_0 = \frac{b_1 rK}{p_2},$$

$$\delta_1 = \frac{d_1}{b_1 p_1}$$

$$\delta_2 = \frac{d_2}{b_2 p_2}$$

$$\sigma_1 = \frac{s_1 Y_0}{b_1 p_1}$$

$$\sigma_2 = \frac{s_2 Z_0}{b_2 p_2}$$

$$\zeta = \frac{b_1 p_1}{r}$$

$$\varepsilon = \frac{b_1 p_1}{b_2 p_2}$$

The new dimensionless equations now become:

$$\zeta \dot{x} = x\left(1 - x - \frac{y}{\beta_1 + x}\right) \qquad (4)$$

$$\dot{y} = y\left(\frac{x}{\beta_1 + x} - \delta_1 - \sigma_1 y - \frac{z}{\beta_1 + x}\right) \qquad (5)$$

$$\dot{z} = \epsilon z\left(\frac{y}{\beta_2 + y} - \delta_2 - \sigma_2 z\right) \qquad (6)$$

3.3 Time series and phase plots

The time series and phase plot chaotic attractors for the continuous system (4) – (6) are given in Figure 3.1 for parameter values, $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2, \delta_2 = 0.18, \sigma_1 = 0, \sigma_2 = 0.1$ & $\epsilon = 0.375$.

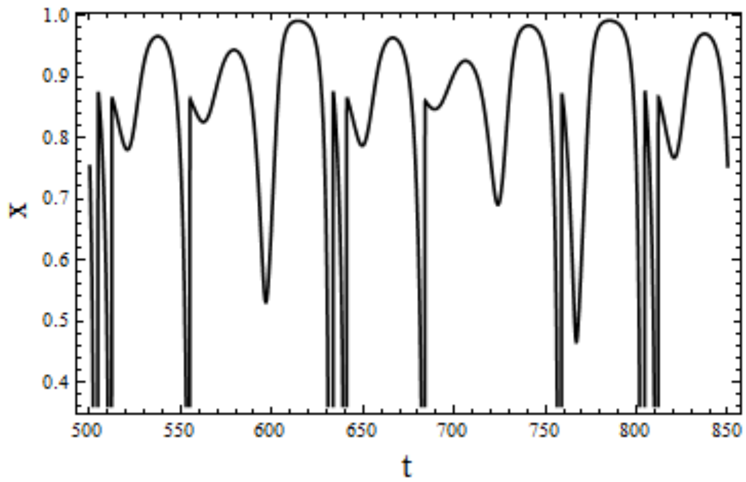


Fig.3.1(a): First plot for time series for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\mathcal{E} = 0.375$.

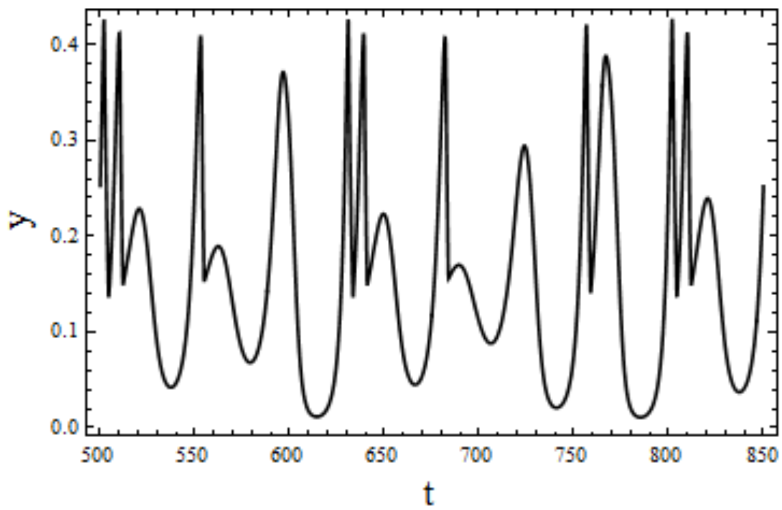


Fig.3.1(b): Second plot for time series for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\mathcal{E} = 0.375$.

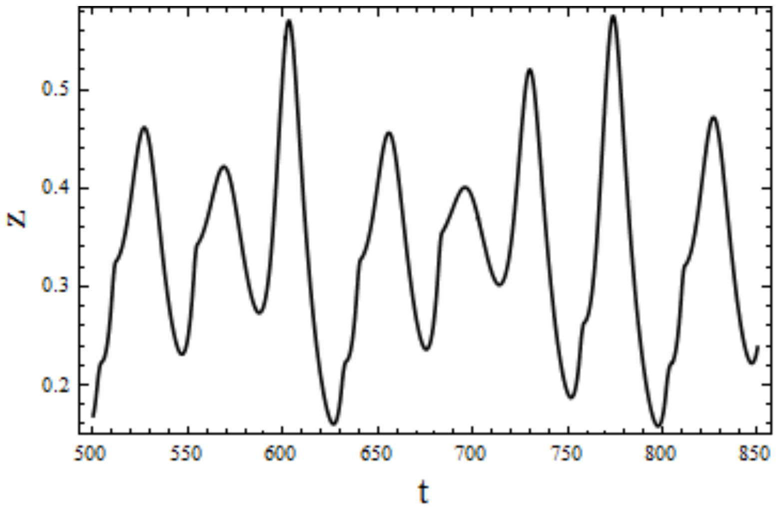


Fig.3.1(c): Third plot for time series for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\mathcal{E} = 0.375$.

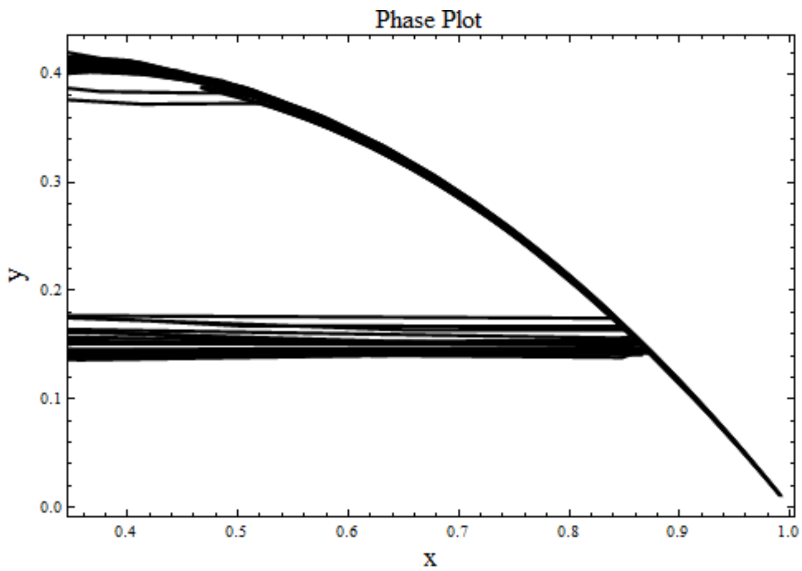


Fig.3.1(d): First phase plot for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\mathcal{E} = 0.375$.

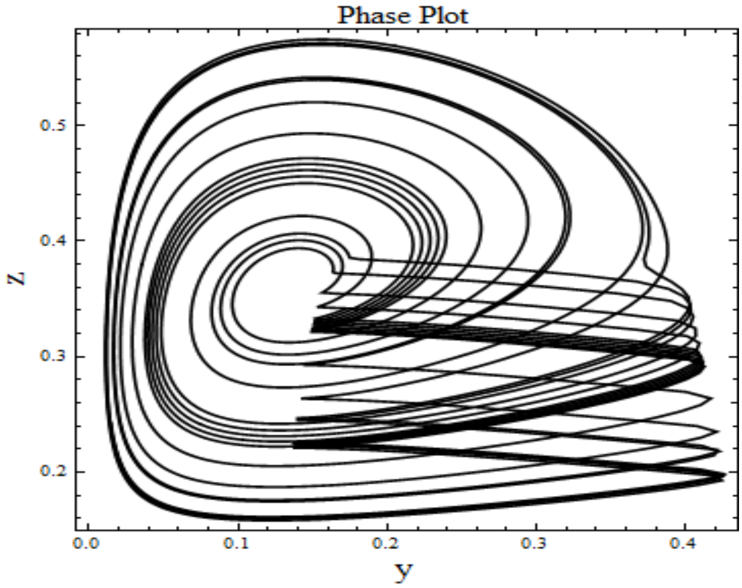


Fig.3.1(e): Second phase plot for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\varepsilon = 0.375$.

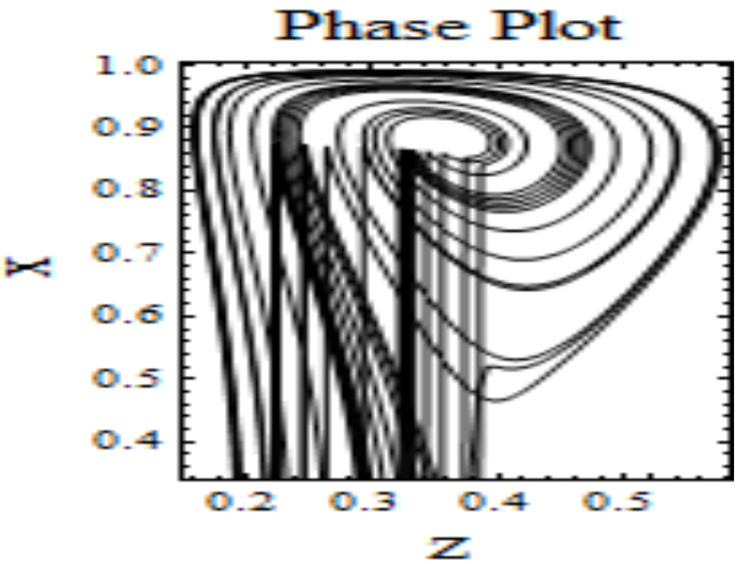


Fig.3.1(f): Third phase plot for chaotic motion when $\zeta = 0.02$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0$, $\sigma_2 = 0.1$ and $\varepsilon = 0.375$.

The continuous system's regular motion can be observed when we increase parameter ζ keeping other parameters intact. For $\zeta = 0.7$, the above type of plot shows regular motion. The time series and phase plots in this regard are given below:

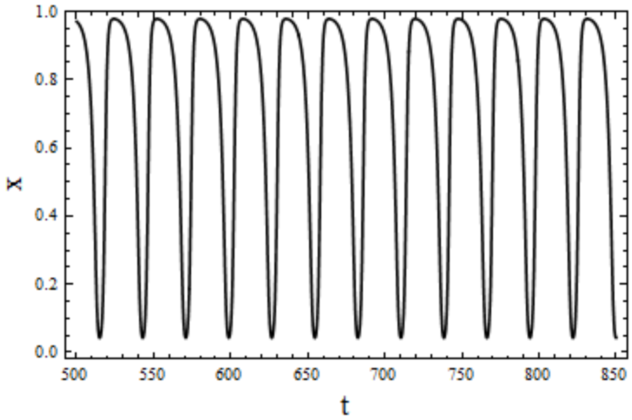


Fig.3.2(a) First time series plot for regular motion when ζ = 0.7.

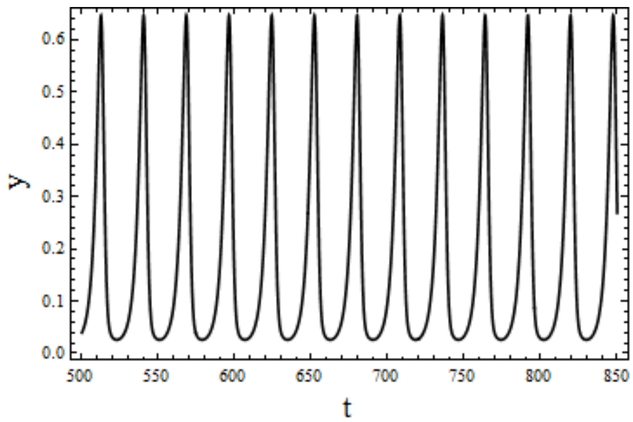


Fig.3.2(b) Second plot for time series for regular motion when ζ = 0.7.

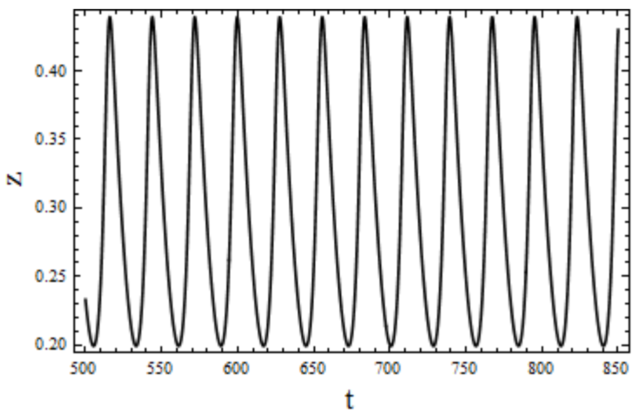


Fig.3.2(c) Third plot for time series for regular motion when ζ = 0.7.

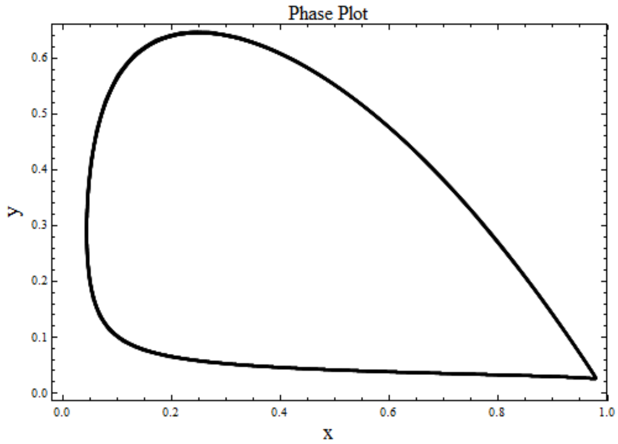


Fig.3.2(d) First phase plot for regular motion when ζ = 0.7.

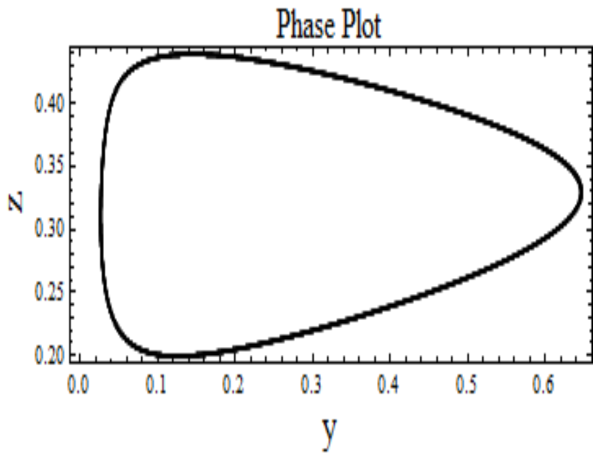


Fig.3.2(e) Second phase plot for regular motion when for ζ = 0.7.

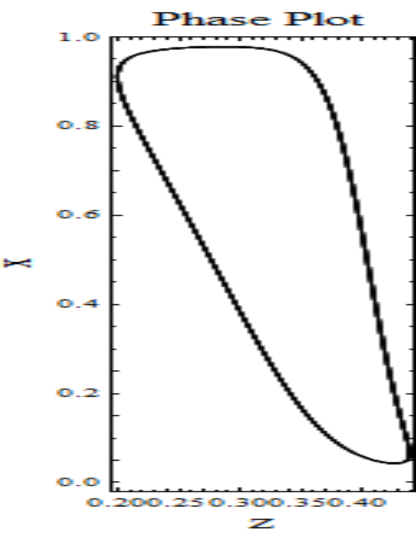


Fig.3.2(f) Third phase plot for regular motion when ζ = 0.7.

Now extending our numerical calculations we transformed the continuous model shown by equations (4) – (6). The continuous model is converted into a discrete model because for studying evolutionary phenomena of real systems, observations are made in discrete time and not continuously. As for example, calculations of populations are done on yearly basis and most of the observational data for experiments are done in discrete time manner at certain intervals. Therefore, changing a continuous model into discrete may be justified in the sense that the results obtained would be more meaningful. Additionally, long term behaviour of a real system can be better analysed if the system is in discrete form. A discrete form of above continuous model can be obtained by Euler's method and written as:

$$x_{n+1}=x_n + \frac{h}{\zeta}x_n(1 - x_n - \frac{y_n}{\beta_1+x_n})$$
$$y_{n+1}=y_n + hy_n(\frac{x_n}{\beta_1+x_n} - \delta_1 - \sigma_1 y_n - \frac{z_n}{\beta_2+y_n})$$
$$z_{n+1}=z_n + h\epsilon z_n(\frac{y_n}{\beta_2+y_n} - \delta_2 - \sigma_2 z_n) \qquad (7)$$

where h must be taken as a very small positive number, i.e. 0 < h << 1. This h would be considered as a parameter of system (7).

3.4 Bifurcation Analysis and attractors

Bifurcation diagrams of discrete food - chain model (7), obtained by varying ϵ, are obtained along all three coordinate planes and shown in Figure 3.3. Parameters are chosen as h=0.001, ζ = 0.01, β_1= 0.26, β_2 = 0.5, δ_1= 0.2, δ_2 = 0.18, σ_1= 0, σ_2 = 0.1 and varying ϵ such that $2.5 \leq \epsilon \leq 3.5$.

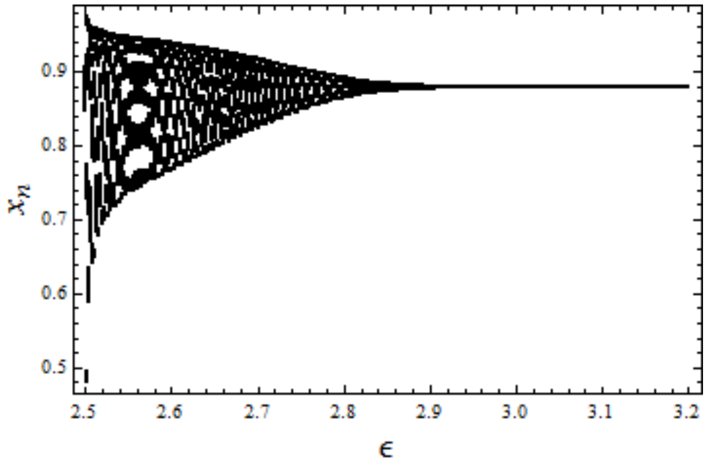


Fig.3.3(a) Bifurcations along x-axis for h=0.001, ζ = 0.01, β_1= 0.26, β_2 = 0.5, δ_1= 0.2, δ_2 = 0.18, σ_1= 0, σ_2 = 0.1 & $2.5 \leq \epsilon \leq 3.5$.

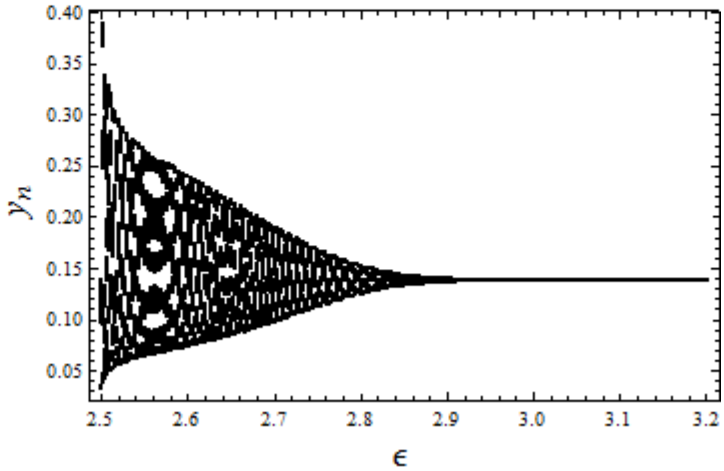


Fig.3.3(b) Bifurcations along y-axis for h=0.001, ζ = 0.01, β_1= 0.26, β_2 = 0.5, δ_1= 0.2, δ_2 = 0.18, σ_1= 0, σ_2 = 0.1 & $2.5 \leq \epsilon \leq 3.5$.

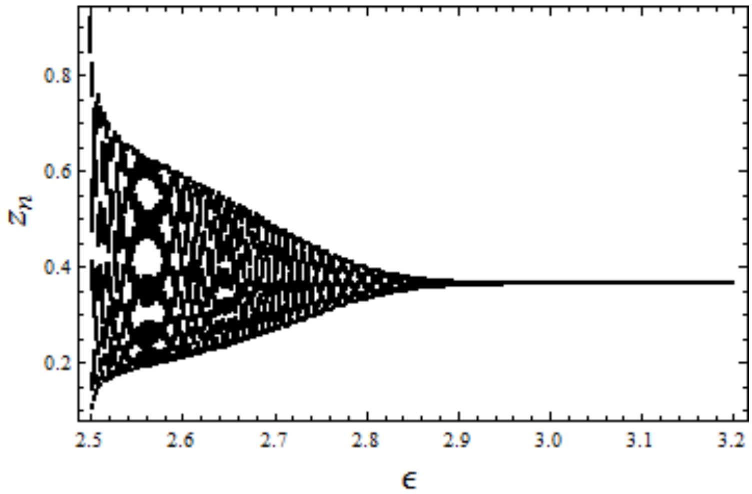


Fig.3.3(c) Bifurcations along z-axis for h=0.001, ζ = 0.01, β_1= 0.26, β_2 = 0.5, δ_1= 0.2, δ_2 = 0.18, σ_1= 0, σ_2 = 0.1 & $2.5 \leq \epsilon \leq 3.5$.

However, by choosing h a greater value, h = 0.01 and keeping other parameters same, the above bifurcation diagrams have certain changes and shown below.

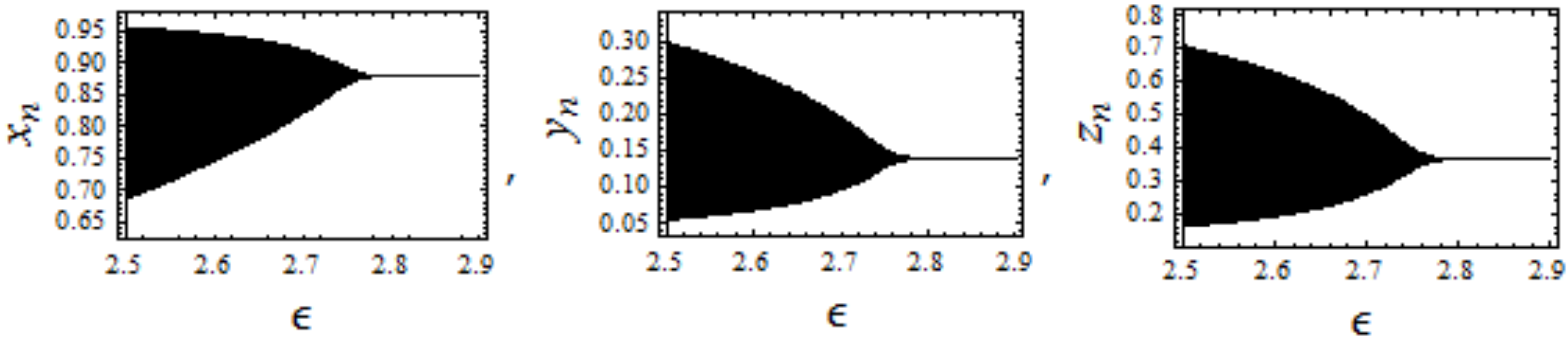


Figure 3.4: Bifurcation diagrams when h = 0.01.

We see from above figures that the motion seems to be chaotic or irregular within $2.5 \leq \epsilon \leq 2.7$.

To obtain regular attractor of the discrete map (7), we have used h=0.01, ζ = 0.01, β_1 = 0.26, $\beta_2 =$ 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0, σ_2 = 0.1and ϵ= 0.05. The figure below is for time series and phase plot in x-y plane. These clearly show that the motion is quasi-periodic and so regular. We proceed our numerical calculations further to obtain the set of parameters for which the evolution of the map (7) is chaotic.

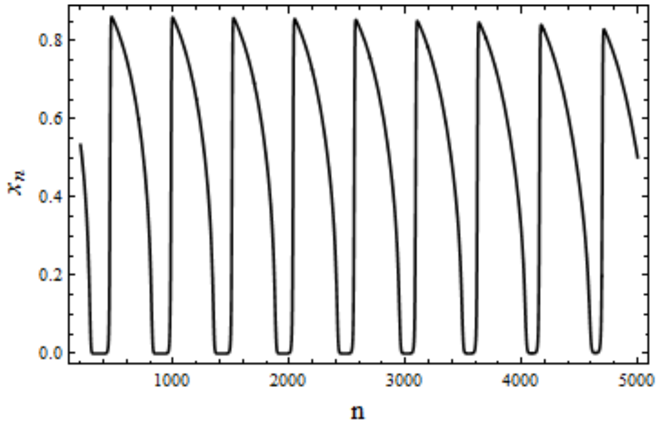


Fig.3.5(a): First chaotic evolution plot when h=0.01, ζ = 0.01, β_1 = 0.26, β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0, σ_2 = 0.1and Ɛ = 0.05.

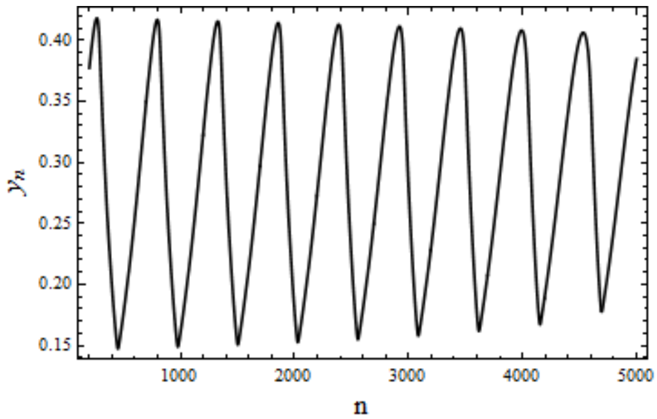


Fig.3.5(b): Second chaotic evolution plot when h=0.01, ζ = 0.01, β_1 = 0.26, β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0, σ_1 = 0.1and Ɛ = 0.05.

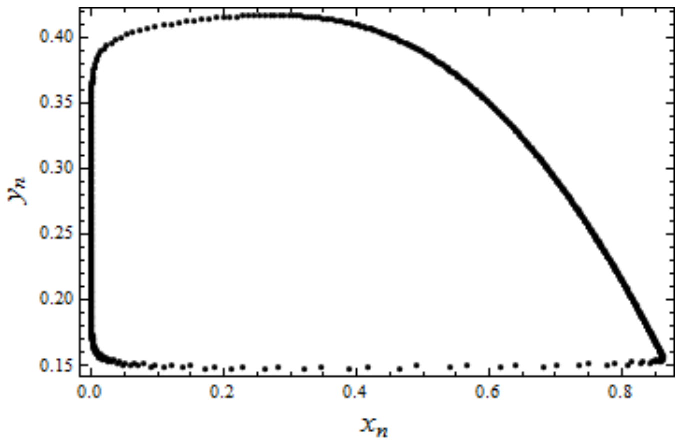


Figure 3.5(c): The x-y phase plot shows that the motion is quasi-periodic type.

3.5 Numerical Simulations for Complexity: Calculations of Lyapunov Exponents, Topological Entropy and Correlation Dimensions.

The complexity of a physical system or a dynamical process expresses the degree to which components engage in organized structured interactions. High complexity is achieved in systems that exhibit a mixture of order and disorder (randomness and regularity) and that have a high capacity to generate emergent phenomena. Complexity in a deterministic dynamical system means certain features that strongly related to the nonlinearity of the system. In this regard mathematics has the largest role in contribution to the study of complex systems leading to the discovery of chaos in deterministic systems during long term evolution. Such systems comprise of many interacting parts and can generate a new quality of collective behaviour through self-organization, e.g. the spontaneous formation of temporal, spatial or functional structures. These systems are often characterized by extreme sensitivity to initial conditions as well as emergent behaviour that are not readily predictable or even completely deterministic. These concepts can be viewed through the calculations for Lyapunov exponents, topological entropies and correlation dimensions.

3.5.1 Lyapunov Exponents (LCEs):

Lyapunov exponents are dynamical measures capable of characterizing deterministic chaos in the system which features to the highly sensitive dependence on initial conditions. Actually, it means the exponential divergence of orbits originated closely with very small difference in initial conditions. Calculation of Lyapunov Exponents is an important and effective element to identify regularity and chaos in the system and can be explained in the following ways:

From above definition, a clear interpretation for Lyapunov exponent is given as: it is the measure of loss of information during the process of iteration.

For a n dimensional system, Lyapunov exponents can be defined by the expression

$$\lambda(X_0, U_0) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \coprod_{t=0}^{n-1} J(X_t) U_0 \right\|, \qquad (8)$$

where $X \in R^n$, $F: R^n \rightarrow R^n$, $U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F and

$$\left\| X_n - Y_n \right\| \approx e^{\lambda(X_0, U_0)n},$$

X_0 and Y_0 are initial points of two trajectories which are supposed to be very close to each other.

Taking h=0.01, $\zeta = 0.01$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0.1$, $\sigma_2 = 0.05$ and ϵ= 0.05,0.1,0.2,0.3,0.4,0.5 and 0.6 we have calculated Lyapunov exponents. Plots of values of LCEs are shown in the figure below.

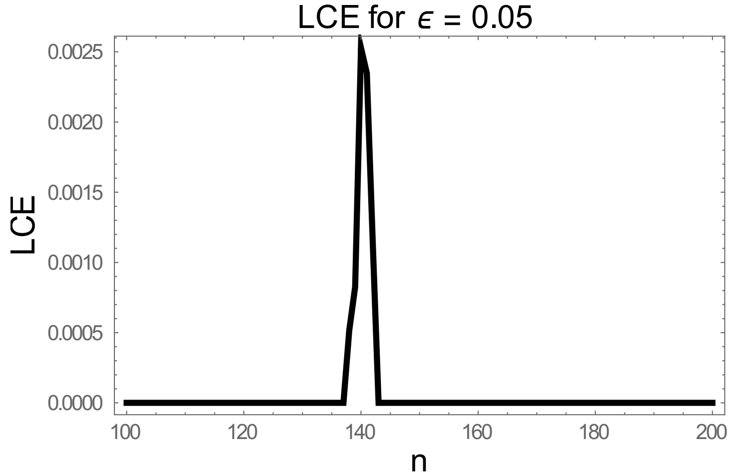


Fig.3.6(a): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.05.

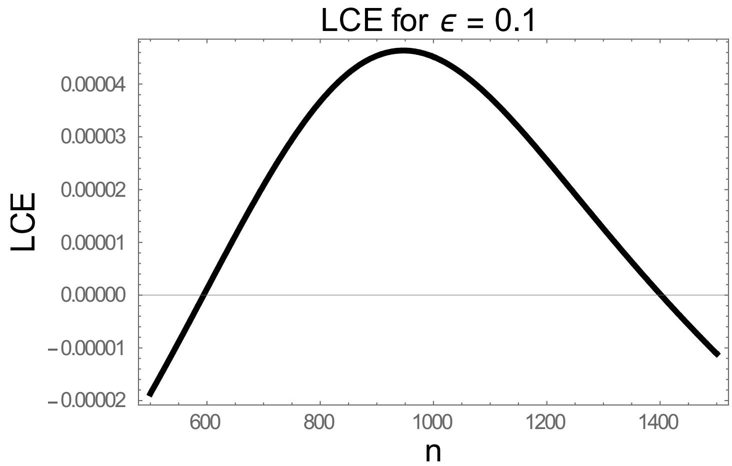


Fig.3.6(b): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.1.

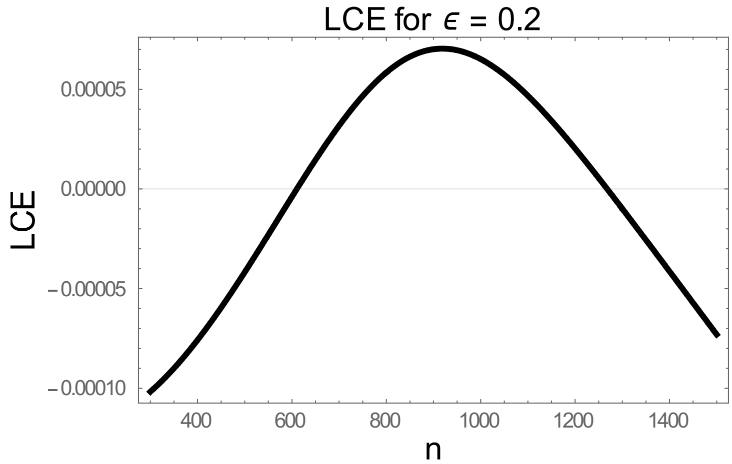


Fig.3.6(c): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.2.

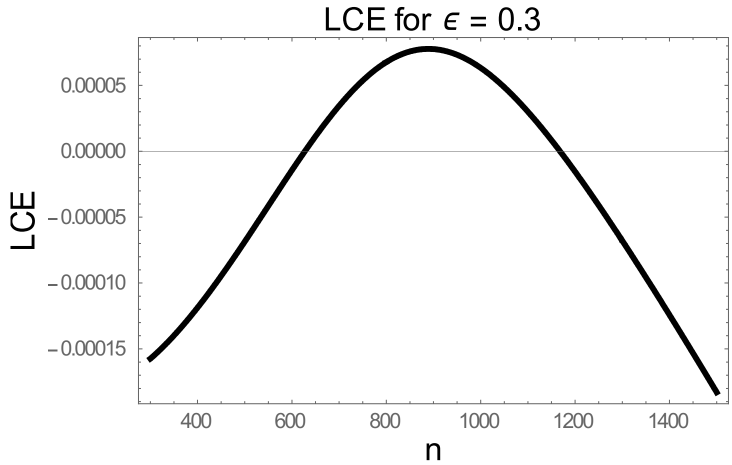


Fig.3.6(d): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.3.

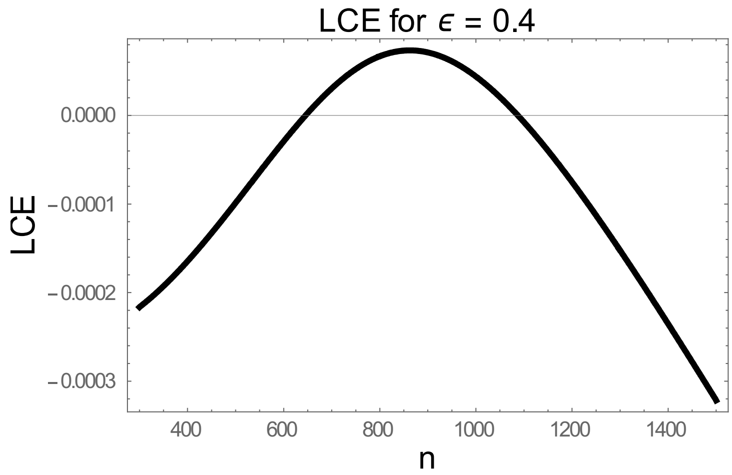


Fig.3.6(e): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.4.

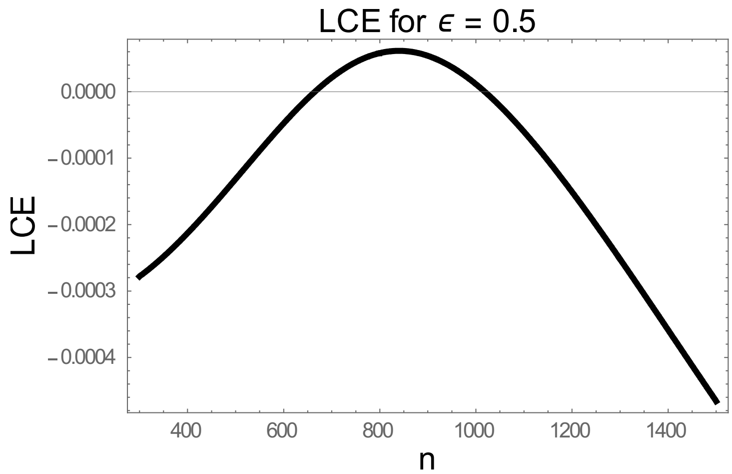


Fig.3.6(f): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.5.

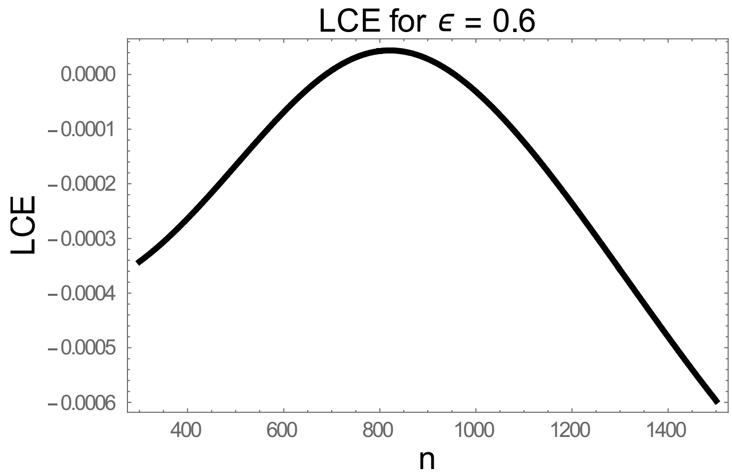


Fig.3.6(g): LCE's with h=0.01, ζ = 0.01, β_1 = 0.26,β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 & ϵ= 0.6.

Study of above plots reveals the fact that for ϵ = 0.05, 0.1, 0.2, 0.3, 0.5 system evolution becomes chaotic but when Ɛ further increases, e.g. ϵ = 0.6, LCEs are negative and system returns to regularity. This type of situation may arise when we choose other sets of parameters.

Extending the numerical simulations further one obtains the plots for Lyapunov exponents (LCE) for the chaotic motion as shown in Figure 3.7.

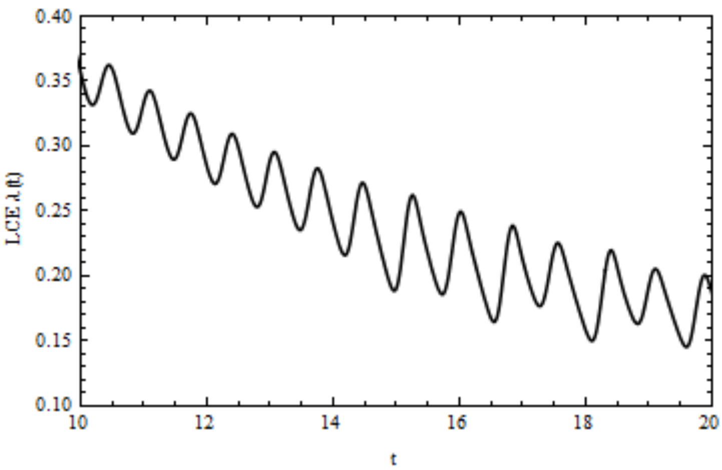


Fig.3.7: Plot of Lyapunov exponents for the chaotic motion.

3.5.2 Topological Entropy

Chaotic attractors of a dynamical system are composed of complex pattern and as the Lyapunov exponent has certain limitations as mentioned in chapter 2. Hence,.

to investigate the chaotic behaviour of any system in a broader spectrum, a better acceptable indicator is the topological entropy. Actually, the topological entropy indicates the complexity of the system. Definition and mathematical derivation of topological entropy was given by Bowen (Bowen, 1973), Gribble (Gribble, 1995), Nagashima and Baba (Nagashima and Baba, 1998) and in the articles by Saha and Kumra (Saha and Kumra, 2013).

Taking h=0.01, ζ = 0.01, β_1 = 0.26, β_2 = 0.5, δ_1 = 0.2, δ_2 = 0.18, σ_1 = 0.1, σ_2 = 0.05 and varying ϵ one can observe easily the evolution is chaotic when ζ = 0.02 and regular when ζ = 5.5. Topological entropy calculations for these two cases have been performed using Mathematica. The plots for chaotic and regular cases are shown, respectively, as plots given below.

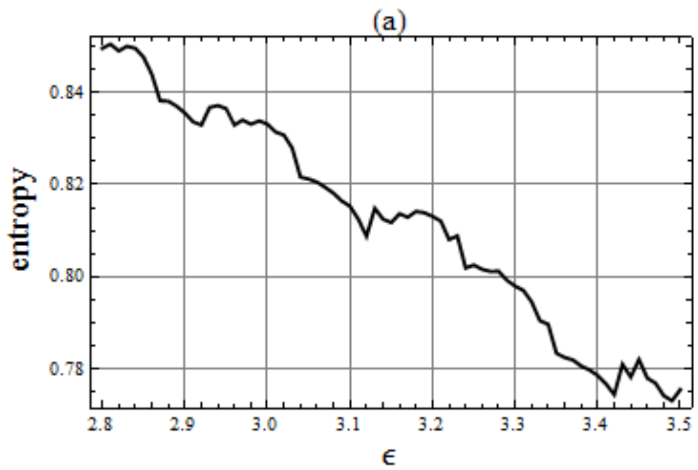


Fig.3.8(a) Topological entropy plots for chaotic evolution ζ = 0.02.

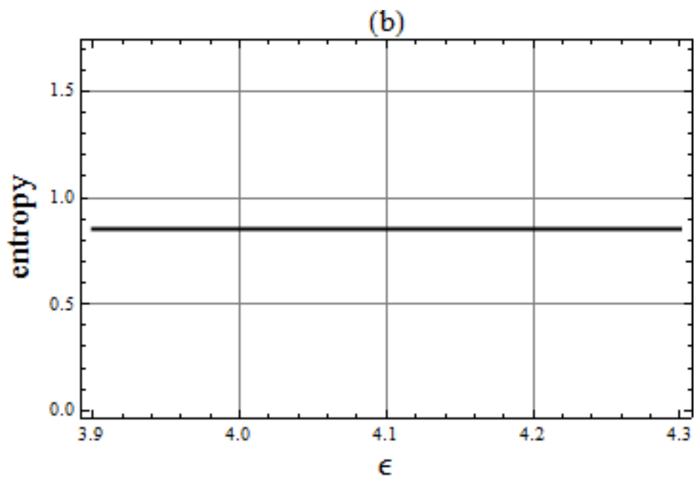


Fig.3.8(b) Topological entropy plots for chaotic evolution ζ = 5.5.

Above plots perfectly provide ideas of complexity in the chaotic system.

3.5.3 Correlation Dimensions

Correlation dimension provides a measure of dimensionality of the chaotic attractor. It is a very practical and efficient method as compared to other methods, like box counting etc. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension also gives a measure of complexity for the underlying attractor of the system. To calculate correlation dimension, one has to use the statistical approach. Here, we use the method described by Martelli (Martelli, 2011).

Correlation dimension provides the dimensionality of the system. Given below we have two plots for correlation integral data for the system for a chaotic case when $\zeta = 0.03$ and for a regular case when $\zeta = 5.5$ while keeping other parameters fixed as h=0.01, $\zeta = 0.01$, $\beta_1 = 0.26$, $\beta_2 = 0.5$, $\delta_1 = 0.2$, $\delta_2 = 0.18$, $\sigma_1 = 0.1$, $\sigma_2 = 0.05$ and $\epsilon = 4.4$, in Figure 3.9.

The plot (b) of the regular case shows zero slope of the curve and zero intercept to y-axis; thus, the correlation dimension is zero in this case. However, plot (a) of the chaotic case is different. By using least square linear fit, one obtains the equation of the straight line approximately fitting the curve as

$Y = 0.2954 - 0.3404\ x$

The intercept of this straight line on the y-axis is equal to $0.2954 \approx 0.3$. Thus, the correlation dimension for chaotic attractor form in this case is approximately equal to 0.3.

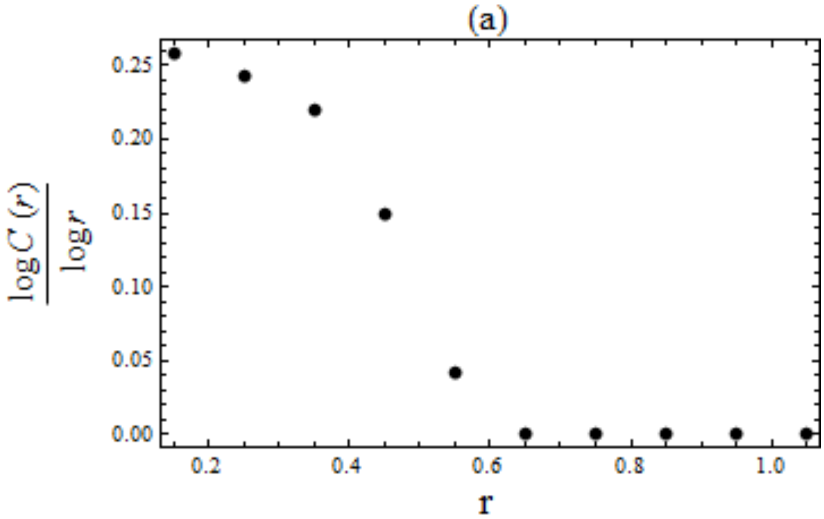


Fig.3.9(a) Correlation dimension for chaotic case when ζ = 0.03.

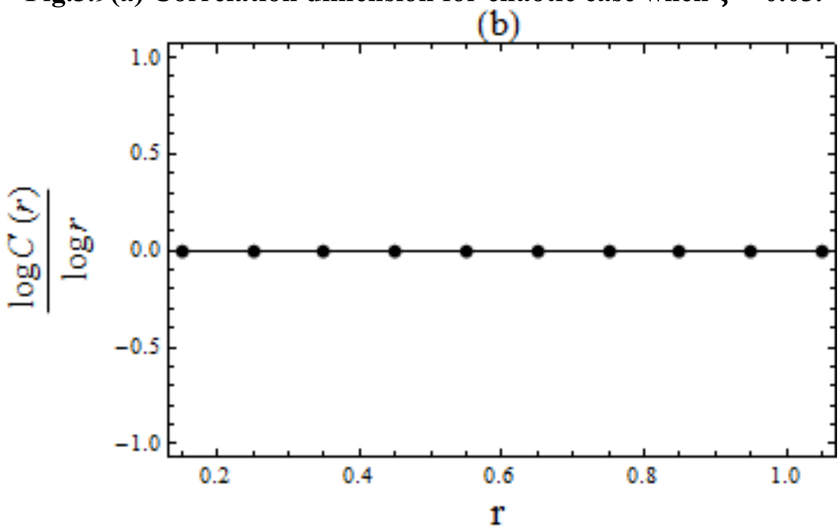


Fig.3.9(b) Correlation dimension for regular case when ζ = 5.5.

3.6 Application of indicator DLI

An indicator named as Dynamic Lyapunov Index (DLI) has recently been introduced by Saha and Budhiraja (Saha and Budhraja, 2007). Its working ability to distinguish between chaotic and regular motions is tested for various discrete systems by Yuasa and Saha (2008), Saha and Tehri (Saha and Tehri, 2010) and Deleanu (Deleanu, 2011). When applied to discretized food chain models and plotted after numerical simulations, it has been observed that DLI's, form a definite pattern for the motion which is regular motion and for chaotic motion it shows randomly distributed points, (with no definite pattern).

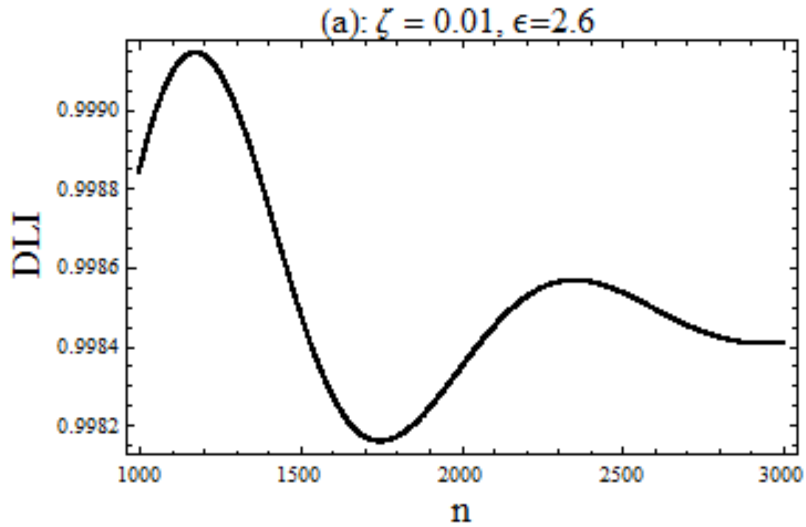


Fig 3.10(a): DLI plots for chaotic evolutions.

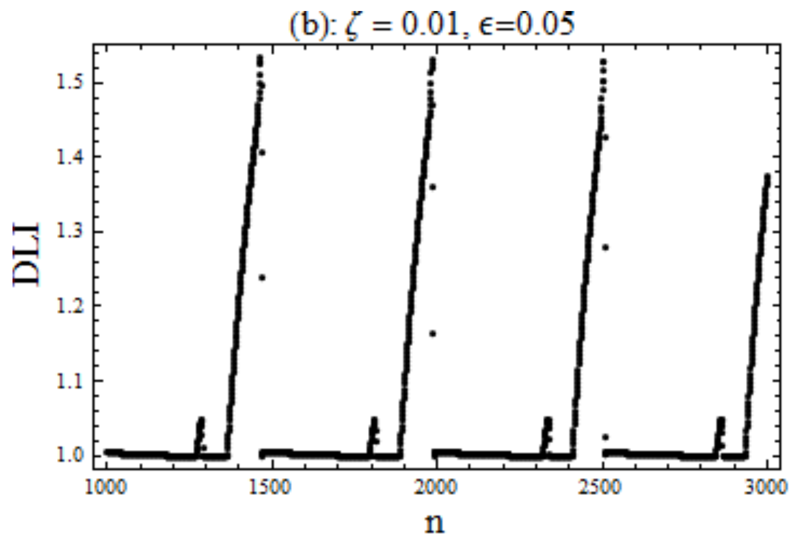


Fig.3.10(b): DLI plots for regular evolutions.

3.7 Discussion

The food chain system introduced by Bo Deng (Deng, 2001), has been considered for the study of complexity arising in the system through some potential numerical simulations like obtaining LCE plots, topological entropies and calculating correlation dimensions for regular as well as for chaotic motions. LCEs provide the measure of exponential divergence, topological entropy is a measure of complexity and correlation dimension is a measure of dimensionality. This last one is nothing but fractal dimension for the chaotic set emerging during evolution. This type of study sheds some light on complexity in the system because nonlinearity plays a big role. Such investigations in food chain system also throw lights on conditions that whether the participating species survives in coexistence or gets extinct. Regularity obtained for certain set of parameters indicates the possibility of survival. Chaotic situations indicate uncertainty which means the possibility of extinction. Finally, we have used an indicator, DLI, described recently for clear identification of regular and chaotic evolution.

Chapter 4

Dynamics of Two-Gene Andrecut-Kauffman System: Chaos and Complexity

4.1 Introduction

Mathematical equations dealing with natural and biological systems are nonlinear in nature and are mostly in complicated form. Nonlinearity can be defined by parameters involved into the system chosen for study. Behaviour of such systems can be understood during evolution by varying parameters under different initial conditions. Computers have added a lot of ease and comfort to the numerical study of this subject by producing many exciting and interesting results. A simple system evolves in simple ways but a complex or complicated system evolves in complicated ways and between simplicity and complexity there cannot be a common ground. Ian Stewart talked about this in his book "Does God play Dice ?" in 1990 (Stewart, 1990) where he included many practical applications of chaos theory. Complex systems have features like cascading failures, far from energetic equilibrium, often exhibiting hysteresis, bistability may be nested, network of multiplicity, emergent phenomena and some more properties. All these are related to the nonlinearity. A systematic evolutionary description and emergence of chaos can be obtained in the beginning chapters of the book edited by Hao-Bai-Lin in 1990 (Lin, 1990). Chaos and irregular phenomena may not require very complicated equations. During evolution, biological systems may display the properties like complexity and chaos. Complexity can be viewed via its systematic nonlinear properties and it is due to the interaction among multiple agents within the system. This has been shown by L M Saha in 2016 (Saha and Kumra, 2016) . Chaotic systems display varied forms of attractors, depending on different sets of parameter values. Complexity and chaos observed in a system can be well understood by measuring elements like Lyapunov exponents (LCEs), topological entropies (Bowen, 1973) correlation dimension etc. Topological entropy is a non-negative number that provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy signifies more complexity. In 1965, Adler, R L Konheim and A G McAndrew (Adler et al., 1965) introduced the notion of entropy as an invariant for continuous mappings. Actually, it measures the exponential growth rate of the number of distinguishable orbits as time advances. In 2007, Andrecut and Kaufmann (Andrecut and Kauffman, 2007) studied the complex dynamics of a discrete model of a two-gene system. They derived the discrete time equations of the system from the chemical reactions corresponding to the gene expression and regulation and showed that the system exhibits regular, chaotic and hyper-chaotic behaviour, depending on the values taken by the control parameters. Since complexity and chaos appear mostly in nonlinear systems, it is necessary to find certain measure of the quantities causing these. Positivity measure of Lyapunov exponents (LCEs) signifies presence of chaos (Abarbanel et al., 1992) and (Sylvia, 2007).Measure of topological entropy signifies the complexity (Baldwin and Slaminka, 1997) and the correlation dimension provides the dimensionality of the attractor of the system (Nagashima and Baba, 1998) and (Grassberger and Procaccia, 1983b) .

While dealing with natural systems, principles of nonlinear dynamics have been extensively used in diverse areas of sciences. In biochemical context nonlinear equations are obtained from chemical reactions appearing in a two-gene model (Andrecut and Kauffman, 2007). Here, chemical reactions are assumed to correspond to gene expression and regulation.

The studies performed in the present article deal with a two-gene Andrecut-Kauffman model (Andrecut and Kauffman, 2007). In this 2-dimensional discrete system, dynamical variables describe the evolution of the concentration levels of transcription factor proteins. To study the characteristics of complex nature of evolutionary phenomena, bifurcation diagrams have been drawn by varying a certain parameter. Then, some numerical investigations are carried forward to obtain Lyapunov exponents (LCEs), topological entropies and correlation dimensions for different sets of parameters of the system. Results obtained are shown through graphics. Finally, the complex nature of evolutions has been discussed on the basis of results obtained through this study.

4.2 The Model

In the present study, we consider a two-dimensional map proposed by Andrecut and Kaufmann (Andrecut and Kauffman, 2007). The map was used to investigate the dynamics of two-gene model for chemical reactions corresponding to gene expression and regulation. The discrete dynamical variables, denoted by x_n and y_n, describe the evolutions of the concentration levels of transcription factor proteins. The map is given by following pair of difference equations:

$$x_{n+1} = \frac{a}{1+(1-b)\,x_n^t + b\,y_n^t} + c\,x_n$$

$$y_{n+1} = \frac{a}{1+(1-b)\,y_n^t + b\,x_n^t} + d\,y_n \qquad (1)$$

With parameter values a = 25, b = 0.1, c = d = 0.18 and t = 3, one obtains four different fixed points with coordinates (2.30409, 2.30409), (- 2.52688, 2.44162), (2.44162, -2.52866), (- 2.39464, - 2.39464) and all are unstable.

For c ≠ d, and when a = 25, b= 0.1, c = 0.18, d = 0.42, and t = 3, again, four unstable fixed points are obtained as (2.2832, 2.5413), (- 2.5458, 2.6566), (2.4613, - 2.7288), (- 2.3744, -2.61705). Therefore, for all these cases, the orbit with an initial point taken nearby any of the fixed points may be unstable and may be chaotic also.

We intend to investigate certain dynamic behaviours of system (1) for cases when c = d and when c ≠ d for evolutions showing irregularities due to presence of chaos and complexity.

4.3 Numerical simulations

4.3.1 Bifurcations

Performing various numerical simulations, the dynamics of evolution have been investigated by obtaining bifurcation diagrams, calculating Lyapunov exponents, topological entropy and correlation dimensions of the system for different cases. For the values of control parameters within the system the following ranges have been proposed:

$a \in [0,50]$, $c \in [-0.4,0.4]$, $b = 0.1$, $d = 0.5$, $t = 3,4,5$.

Case 1: Taking c = d, bifurcation diagrams are drawn along the directions x and y, by varying c for cases t = 3, 4, 5 and certain fixed values of other parameters as shown below in figure 4.1.Then, plots of attractors have been obtained for parameters a = 25, b = 0.1, t = 3 and (i) for regular case c = d = 0.32 and (ii) for chaotic case c = d = 0.18 and shown in figures 4.2 below. In each case when t = 3, 4, 5, bifurcations show period doubling leading to chaos and then to regularity. Also, bistability and folding nature of phenomena are appearing here.

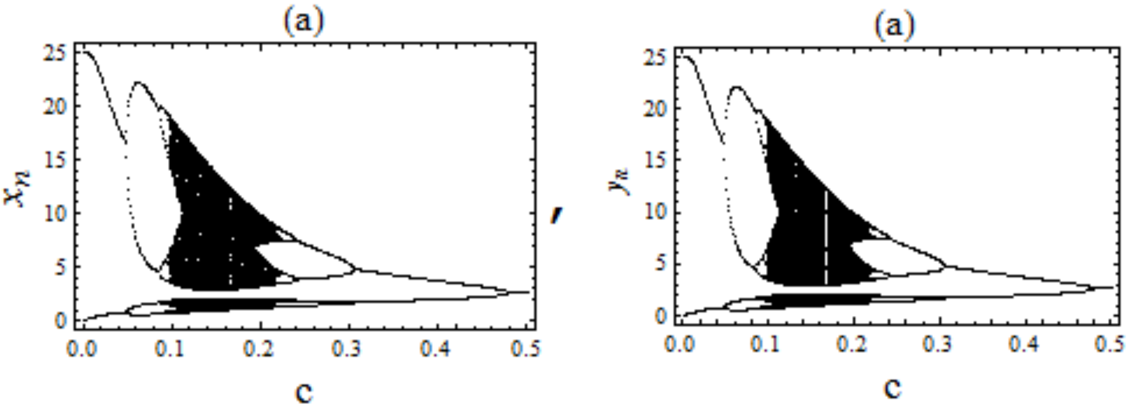


Fig.4.1(a): The bifurcation scenarios for parameters c = d , t = 3, a = 25, b = 0.1 & 0 ≤ c ≤ 0.5.

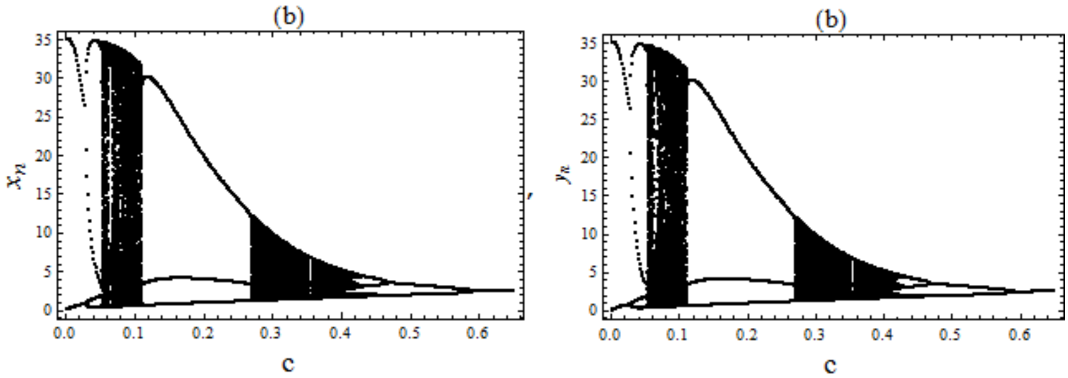


Fig.4.1(b): The bifurcation scenarios for parameters c = d, t = 4, a = 35, b = 0.1 and 0 ≤ c ≤ 0.65.

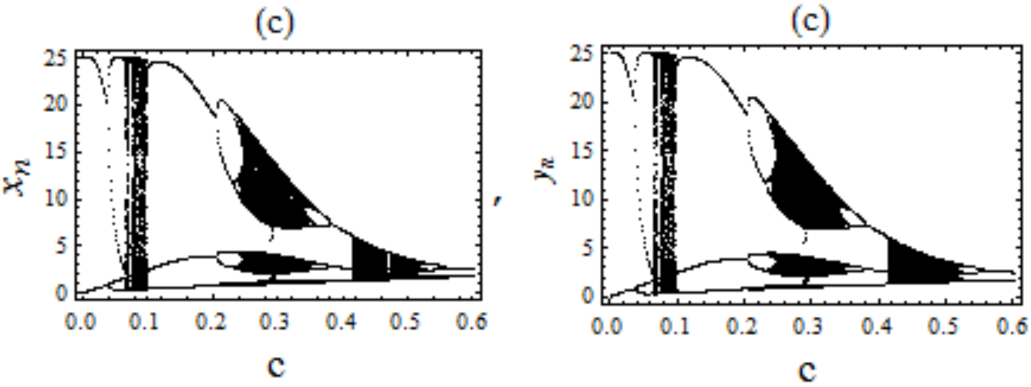


Fig.4.1(c): The bifurcation scenarios for parameters c = d, t = 5, a = 25, b = 0.1 and 0 ≤ c ≤ 0.5.

Case 2: Taking c ≠ d, bifurcation diagrams are drawn along the directions x and y**.**

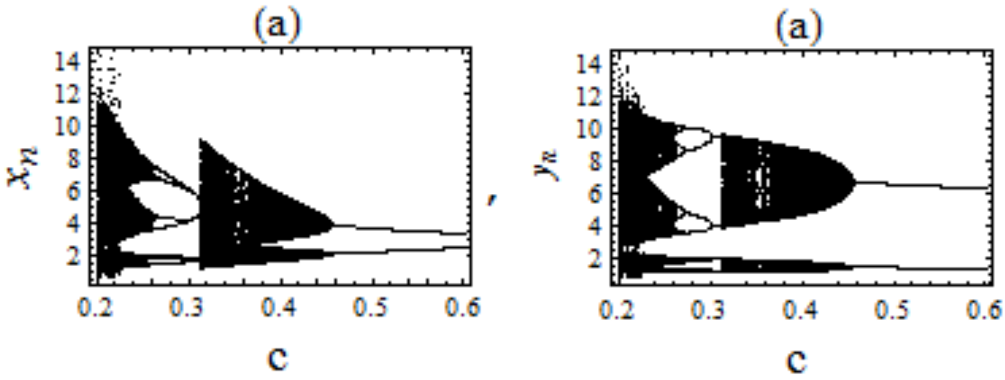


Fig.4.1(d): The bifurcations for the case c≠d,t=3, a = 25, b = 0.1 & d = 0.20.

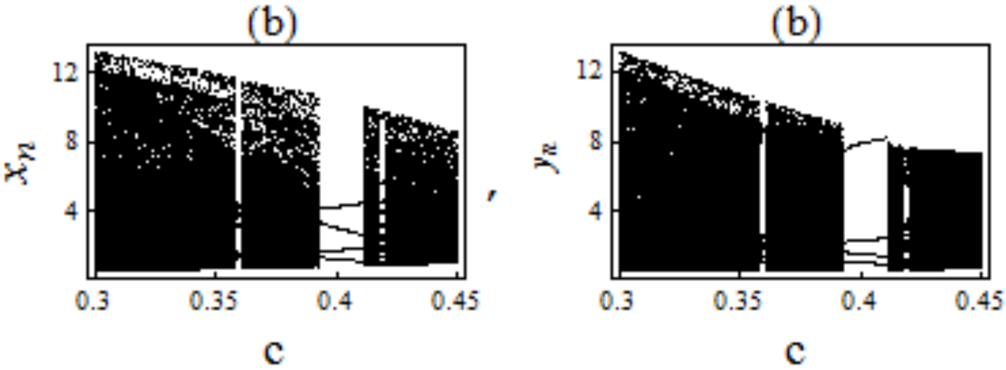


Fig.4.1(e): The bifurcations for the case c≠d,t=4, a = 25, b = 0.1 & d = 0.30.

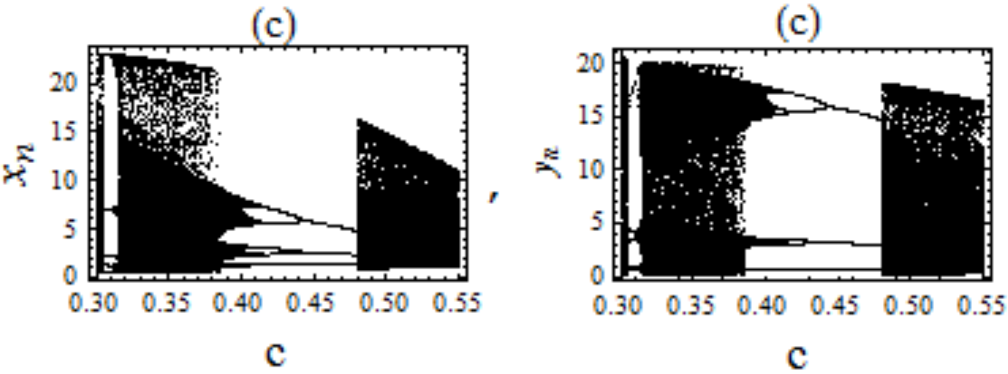


Fig.4.1(f): The bifurcations for the case c≠d ,t=5, a = 25, b = 0.1 & d = 0.20.

4.3.2 Attractors

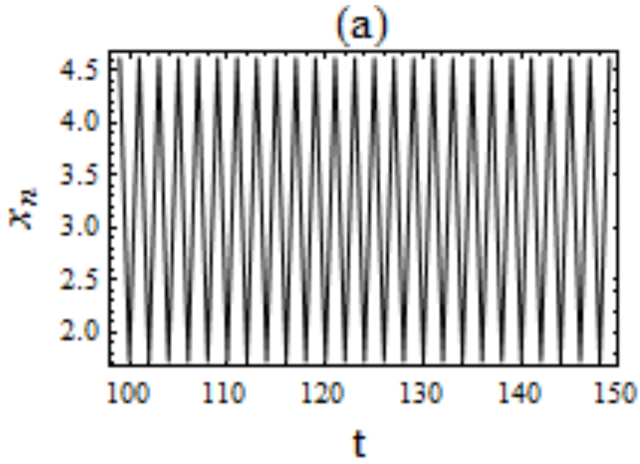


Fig.4.2(a): Time series plot for regular case with a = 25, b = 0.1, t = 3 & c = d = 0.32.

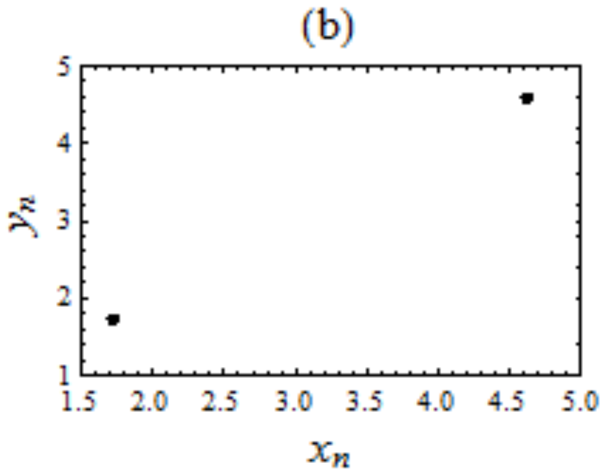


Fig.4.2(b): Phase plane plot for regular case with a = 25, b = 0.1, t = 3 & c = d = 0.32.

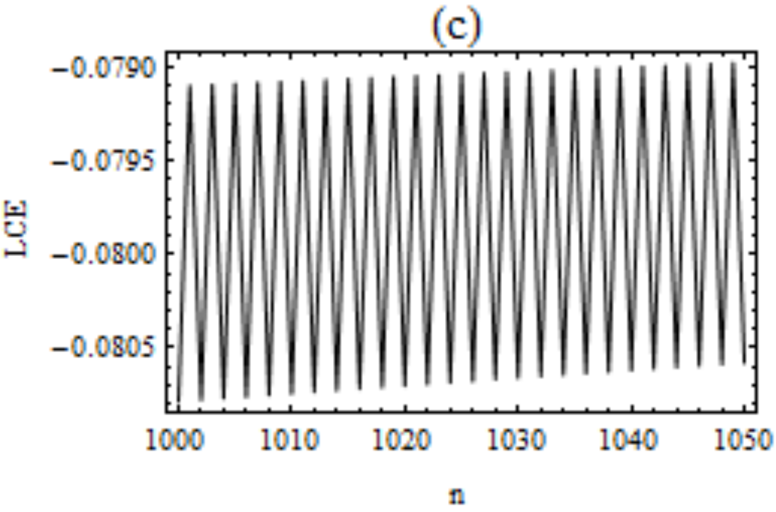


Fig.4.2(c): Lyapunov exponents for regular case with a = 25, b = 0.1, t = 3 & c = d = 0.32.

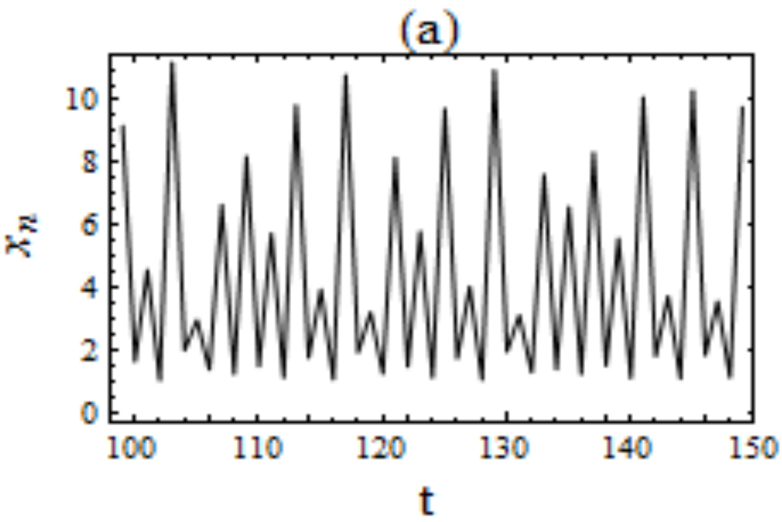


Fig.4.2(d): Time series plot for chaotic case with a = 25, b = 0.1, t = 3 & c = d = 0.18.

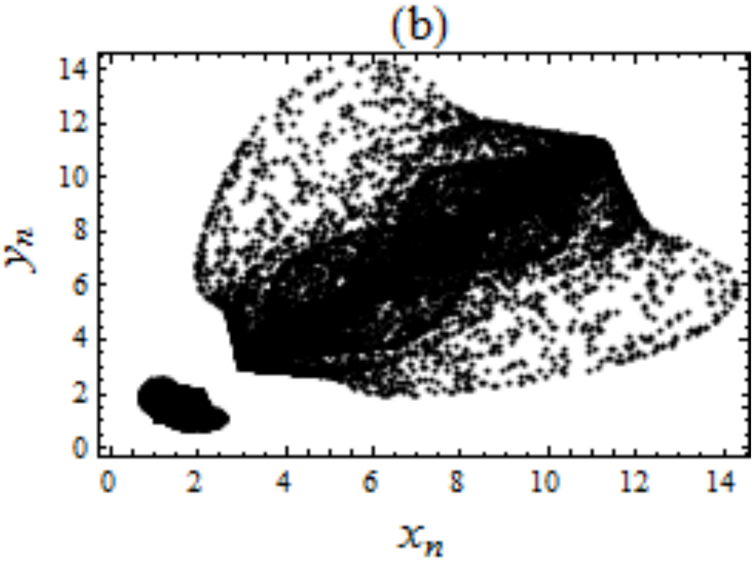


Fig.4.2(e): Phase plane plot for chaotic case with a = 25, b = 0.1, t = 3 & c = d = 0.18.

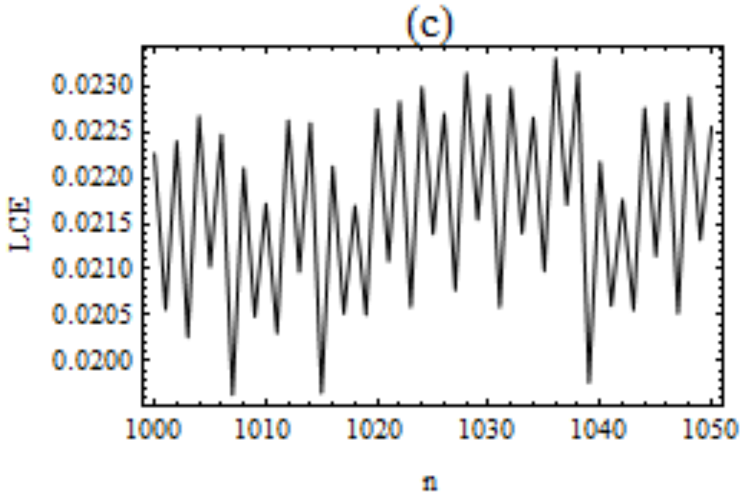


Fig.4.2(f): Lyapunov exponents for chaotic case with a = 25, b = 0.1, t = 3 & c = d = 0.18.

4.3.2 Lyapunov Exponents and Topological Entropies

For chaotic evolution, when a = 25, b = 0.1, t = 3, c = d = 0.18, Lyapunov exponents are obtained and their plots are shown in Fig.4.3. Numerical investigations are further proceeded for the calculation of topological entropies. In Fig.4.4, plots of topological entropies are presented for t = 3, 4, 5 and for different ranges of parameter c. Analysis of these plots, gives an impression that for the case t = 3, system shows enough complexity in the range $0.05 \leq c \leq 0.23$.

For the case t = 4, the system shows high complexity in the range $0 \leq c \leq 0.22$ and in case t = 5, high complexity appears in the range $0 \leq c \leq 0.44$.

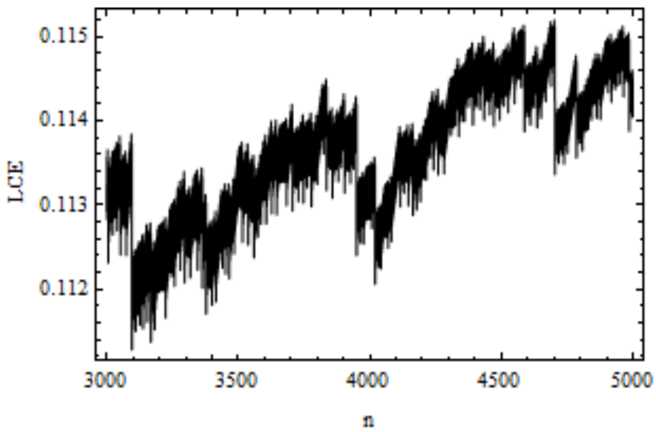


Fig.4.3(a): First plot for Lyapunov exponents for chaotic evolution with a = 25, b = 0.1, t = 3, c = d = 0.18.

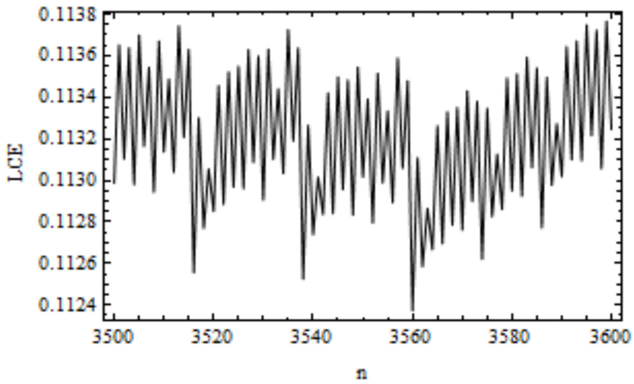


Fig.4.3(b): Second plot for Lyapunov exponents for chaotic evolution with a = 25, b = 0.1, t = 3, c = d = 0.18.

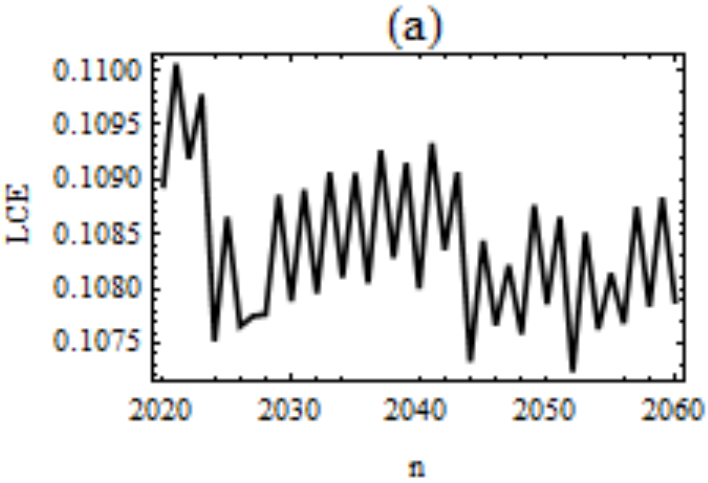


Fig.4.3(c): LCE's for t=3, a = 25, b = 0.1, c = 0.28, d = 0.12.

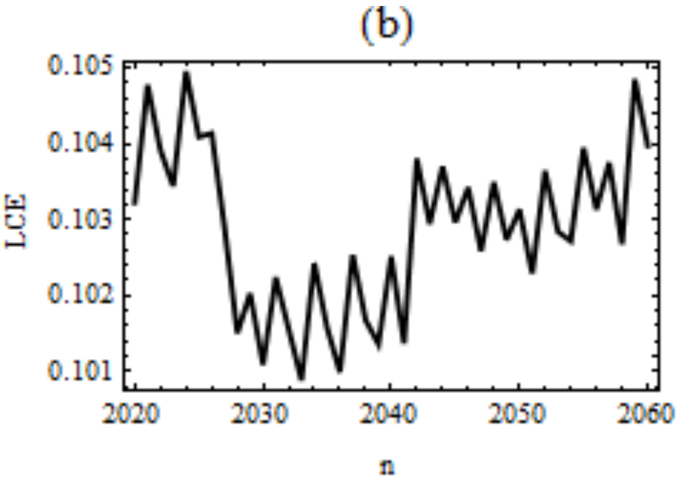


Fig.4.3(d): LCE's for t=4, a = 25, b = 0.1, c = 0.2, d = 0.15.

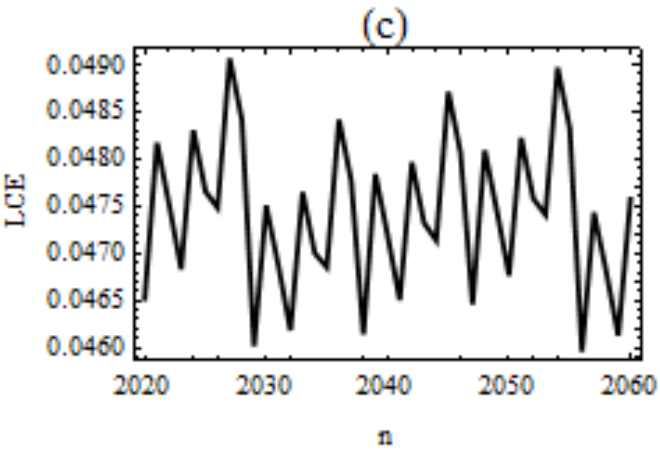


Fig.4.3(e): LCE's for t=5, a = 25, b = 0.1, c = 0.2, d = 0.15.

4.3.3 Topological Entropy

The topological entropy is shown via following graphics in figure 4.4.

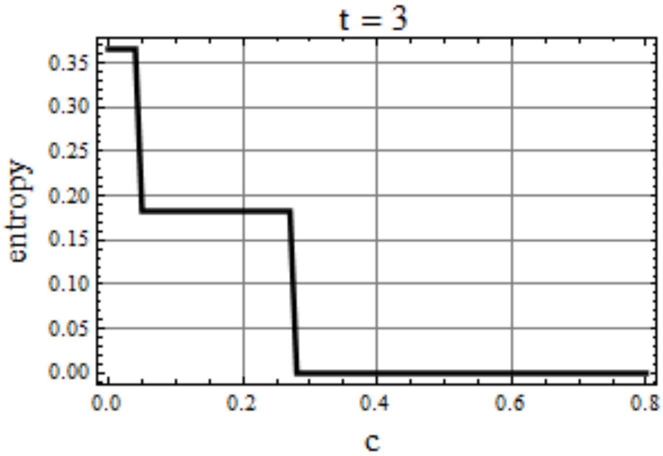


Fig.4.4(a): Topological Entropy when c = d ,t = 3, a = 25, b = 0.1 & 0 ≤ c ≤0.5.

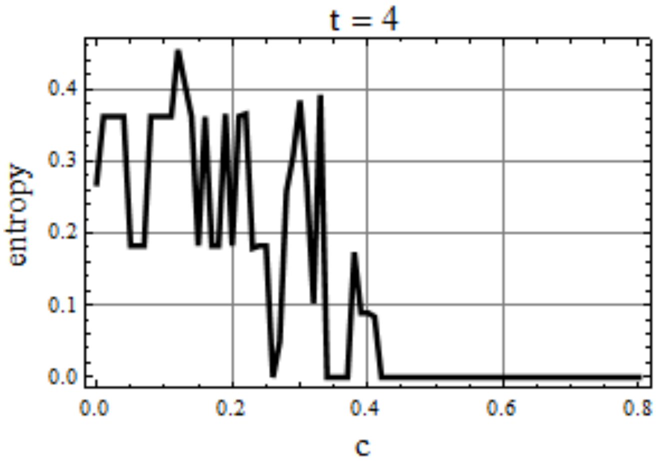


Fig.4.4(b): Topological Entropy when c = d, t = 4, a = 35, b = 0.1 & 0 ≤ c ≤ 0.65.

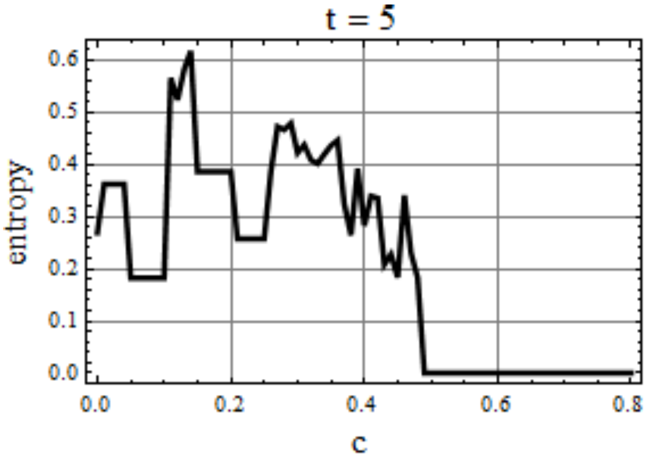


Fig.4.4(c): Topological Entropy when c = d, t = 5, a = 25, b = 0.1 & 0 ≤c ≤0.8.

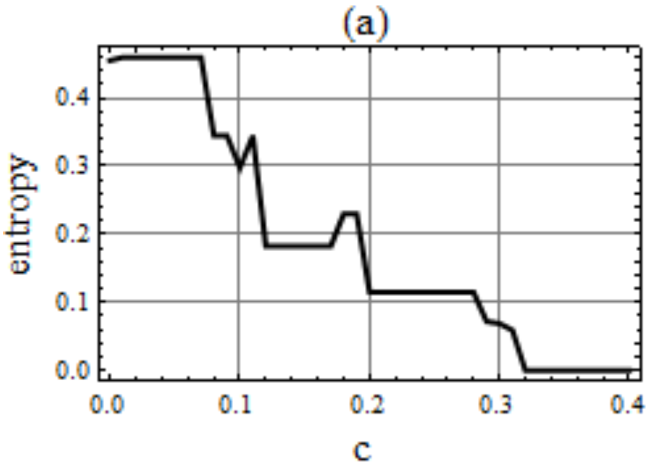


Fig.4.4(d): Topological Entropy when t=3, a = 25, b = 0.1, d = 0.15.

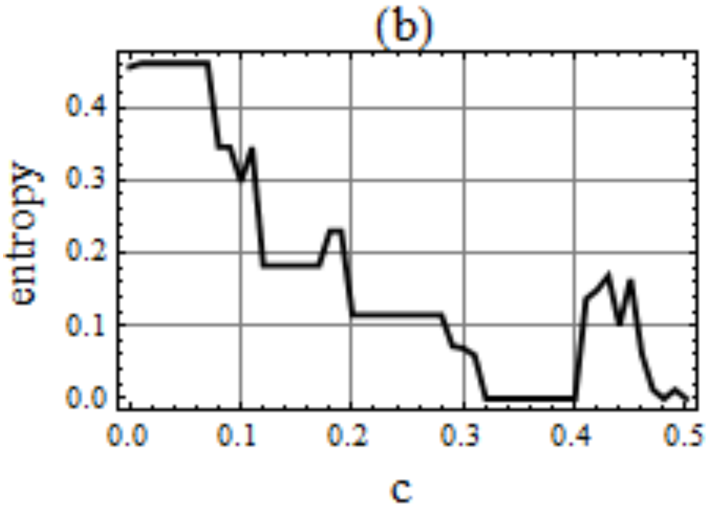


Fig.4.4(e): Topological Entropy when t=4, a = 25, b = 0.1, d = 0.15.

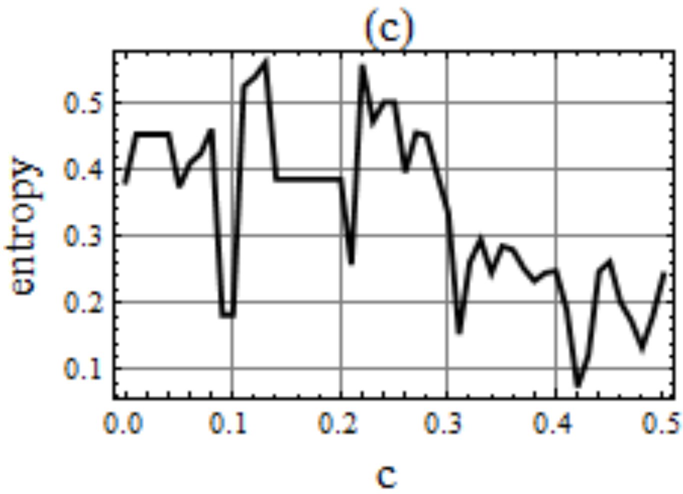


Fig.4.4(f): Topological Entropy when t=5, a = 25, b = 0.1, d = 0.15.

When parameters c and d both were allowed to vary, one gets 3D plots for topological entropies as shown here in Fig.4.5.

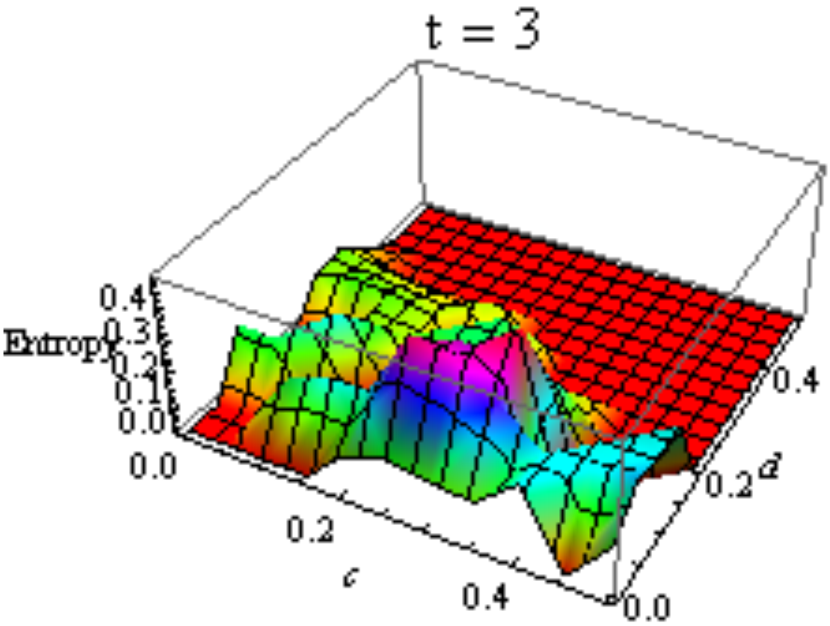


Fig.4.5(a): 3-D plot of Topological Entropy for t=3, a = 25, b = 0.1, 0 ≤c ≤ 0.5 & 0 ≤ d ≤ 0.5.

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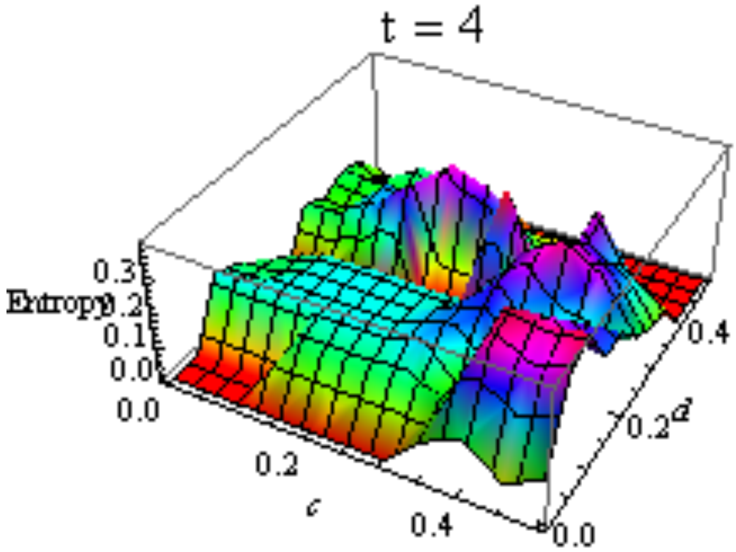

Fig.4.5(b): 3-D plot of Topological Entropy for t=4, a = 25, b = 0.1, 0 ≤c ≤ 0.5 & 0 ≤ d ≤ 0.5.

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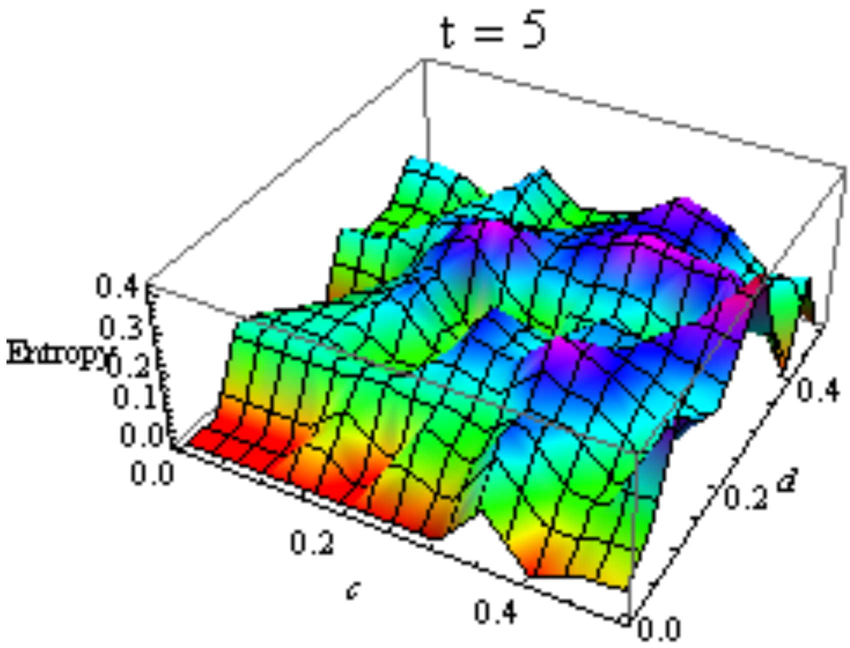

Fig.4.5(c): 3-D plot of Topological Entropy for t=5, a = 25, b = 0.1, 0 ≤c ≤ 0.5 & 0 ≤ d ≤ 0.5.

4.3.4 Correlation Dimensions

Correlation dimension gives the measure of dimensionality. Chaotic evolutions in dynamical systems are characterized by a chaotic set, "strange attractor", which has fractal structure. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. A statistical method can be used to determine correlation dimension. It is an efficient and practical method in comparison to other methods, like box counting etc. The procedure to obtain correlation dimension follows from some steps calculations performed in a previous study (Grassberger and Procaccia, 1983b, Grassberger and Procaccia, 1983a). We have also discussed these steps in detail in the chapter 2 of this thesis.

Extending further the numerical study, correlation dimensions of system (1) have been calculated for various chaotic cases discussed above. For this the method used is that of Martelli with Mathematica codes (Nagashima and Baba, 1998). In brief, the method can be described as follows:

Consider an orbit $O(x_1) = \{x_1, x_2, x_3, x_4, \ldots\ldots\}$, of a map $f: U \rightarrow U$, where U is an open bounded set in R_n. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral,

$$C(r) = \lim_{n \to \infty} \frac{1}{n(n-1)} \sum_{i \neq j}^{n} H\left(r - \left\| x_i - x_j \right\| \right), \qquad (2)$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors x_i and x_j that are closer to r.

The correlation dimension D_c of $O(x_1)$ is then defined as

$$D_c = \lim_{r \to 0} \frac{\log C(r)}{\log r} \qquad (3)$$

To obtain D_c, log C(r) is plotted against log r, shown in Fig.4.6, and then we find a straight line fitted to this curve. The intercept of this straight line on y-axis provides the value of the correlation dimension D_c.

Computation of correlation dimension has been carried out for all the cases described in this article for different set of values of parameters as shown in the following table.

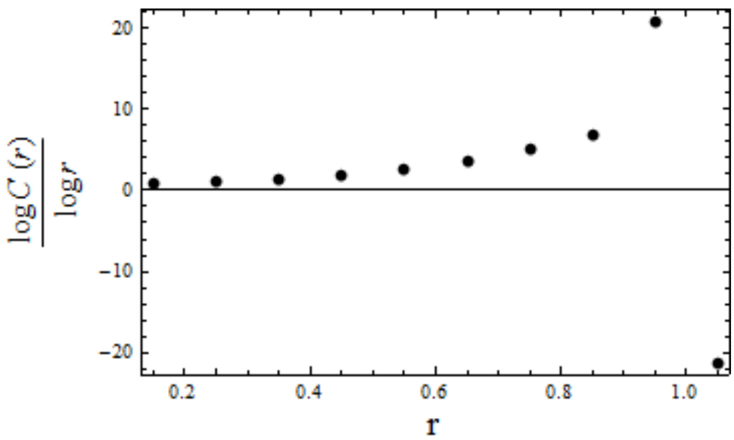

Fig.4.6: Plot of correlation integral curve for the case t = 3. Parameters values are a = 25, b = 0.1, c = 0.28, d = 0.12.

Cases (t) / Parameters	a	b	C	D	Dimension
t = 3	25	0.1	0.18	0.18	3,81869
t = 4	25	0.1	0.18	0.18	3.05258
t = 5	25	0.1	0.18	0.18	3.11754
t = 3	25	0.1	0.28	0.12	3.16201
t = 4	25	0.1	0.28	0.12	3.96724
t = 5	25	0.1	0.28	0.12	4.05859
t = 3	35	0.1	0.2	0.2	3.8041
t = 4	35	0.1	0.2	0.2	3.4164
t = 5	35	0.1	0.2	0.2	4.73368

Table 4.1: Numerical values of correlation dimensions for different values of t.

4.4 Discussion

Two gene Andrecut - Kaufmann system has been studied carefully to understand chaotic phenomena during its evolution together with complexities present in the system. Investigation is made for cases t = 3, 4, 5 only but one can extend it for cases $t \geq 6$ also. Bifurcation plots showed here indicate the phenomena of period doubling and bistability in all these cases. Chaotic evolutions with periodic windows are clearly visible. Presence of complexity in the system can be observed by plots of topological entropies. Variations of topological entropies can be observed in 3D plots shown in figures. Numerical values of correlation dimensions, shown in Table 4.1, provide approximate dimensionality of chaotic attractors.

Chapter 5

Complexities in a Plant-Herbivore Model

5.1 Introduction

The beauty of nature surrounding us lies in its complexity. It is indeed very interesting to study this complexity and infer exciting results from calculations. Complex systems have features like cascading failures, far from energetic equilibrium, often exhibit hysteresis, bistability may be nested, networks of multiplicity, emergent phenomena and some more properties. Complexity can be viewed via its systematic nonlinear properties and it is due to the interaction among multiple agents within the system. A complex system is made up of a large number of parts that interact in a non-simple way and complexity is the property of the relationship between a system and various representations of the system. Due to its nonlinear structure, any biological systems may display the properties like complexity and chaos. Elaborate descriptions on complexity can be viewed from some well written articles (Simon, 1962, Weaver, 1948). In 2001, S. M. Manson wrote "Simplifying complexity: a review of complexity theory" where he explained the concepts and principles of complexity in detail (Manson, 2001). Complexity and chaos observed in a system can be well understood by measuring elements like Lyapunov exponents (LCEs), topological entropies, correlation dimension etc. Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. For a system, more topological entropy signifies more complexity. Actually, it measures the exponential growth rate of the number of distinguishable orbits as time advances (Adler et al., 1965, Bowen, 1973). Positivity measure of Lyapunov exponents (LCEs) signifies presence of chaos (Benettin et al., 1980, Abarbanel et al., 1992). Measure of topological entropy signifies the complexity (Adler et al., 1965, Balmforth et al., 1994, Baldwin and Slaminka, 1997) and the correlation dimension provides the dimensionality of the attractor of the system (Grassberger and Procaccia, 1983a).

Plant-herbivore interactions have a very important role in our environment. Since plants and associated herbivores constitute more than half of the eukaryotic species inhabiting our world, the understanding of this relationship between animals and plants is extremely important for land management. The removal of a particular plant species or group may result in the disappearance of many animals from an area. This argument provides a strong basis for studying animal plant relationships and infer results for future. Plant-herbivore model is a generalization of the host-parasite and Nicholson-Bailey models, studied under various assumptions and modifications (Harper, 1977, Antonovics and Levin, 1980). More extensive studies have been carried out recently on plant-herbivore model by formulating a suitable mathematical model (Abbott and Dwyer, 2007, Kang et al., 2008) and some interesting facts on evolutionary behavior have been pointed there.

The present work is based on the non-dimensional mathematical model proposed in a recent article (Abbott and Dwyer, 2007). The objectives are to study the deterministic model for chaotic evolution and effect on evolution due to complexities presence in the system. Measures like Lyapunov exponents, topological entropies and correlation dimensions (Saha and Kumra, 2016, Saha et al., 2016) have been used to discuss the evolutionary behavior of the system. In the processes of study, we have discussed the stability criteria of the steady state solution. This is done by drawing of bifurcation diagrams for the model by varying a parameter while keeping others constant and discussion of certain properties of the motion followed by the calculation of LCEs, topological entropies, correlation dimensions and their graphical representation.

5.2 The Model

The discrete 2-D model is controlled by two parameters "r", the nutrient uptake rate of the plant, and "a", the amount of leaves eaten by the herbivore. Mathematical analysis and simulations of this model provide us with biological insights that may be used to devise control strategies to regulate the population of the herbivore. We are focusing our study on a host parasite model where P and H are the population biomasses of the host (a plant) and the parasite (a herbivore) in successive generations n and $n+1$ respectively, (Kang, Armbruster and Kuang,2008). We consider the following non-dimensionalized system :

$$x_{n+1} = x_n \, e^{r(1-x_n)-a y_n}$$

$$y_{n+1} = x_n \, e^{r(1-x_n)} \, (1-e^{-a y_n}) \qquad (1)$$

We model the plant and herbivore dynamics through their biomass changes. We assume that soil acts as an unlimited reservoir for biomass growth. A herbivore in model (1) has a one year life cycle. Without the herbivore, the biomass of the plant population follows the dynamics of the Ricker model with a constant growth rate r and plant carrying capacity P_{max}.

The Ricker dynamics determines the amount of new leaves available for consumption for the herbivore. The parameter a is a constant that correlates the total amount of biomass that an herbivore consumes. To obtain non-dimensional form of the system, we set it as:

$$x_n = \frac{P_n}{P_{max}} \text{ and } y_n = \frac{H_n}{P_{max}} .$$

In the section below, we have done numerical simulations like finding attractors and bifurcation diagrams. We have varied ‘r’ and ‘a’ over different ranges and intervals.

5.3 Numerical Simulations

5.3.1 Bifurcations

The Bifurcation diagrams are displayed below in Fig.5.1

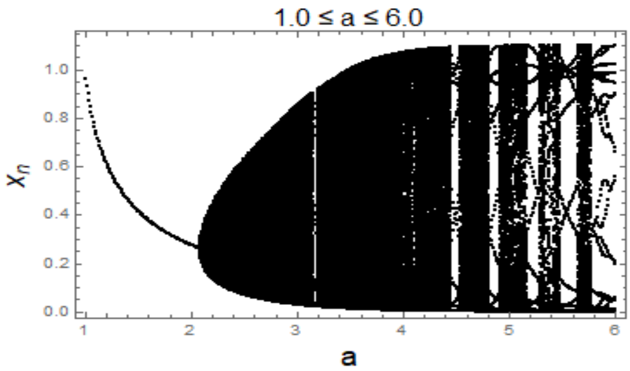

Fig.5.1(a) Bifurcations on x-axis for 1.0≤a≤6.0.

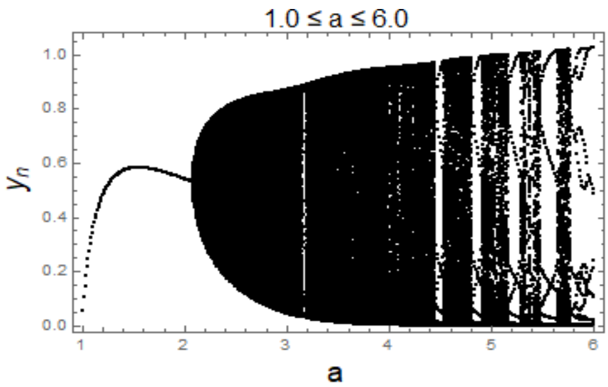

Fig.5.1(b) Bifurcations on y-axis for 1.0≤a≤6.0.

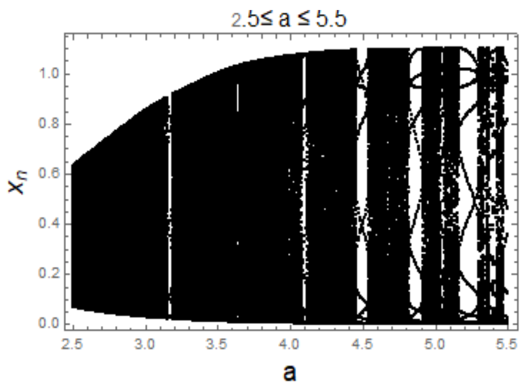

Fig.5.1(c) Bifurcations on x-axis for 2.5≤a≤5.5.

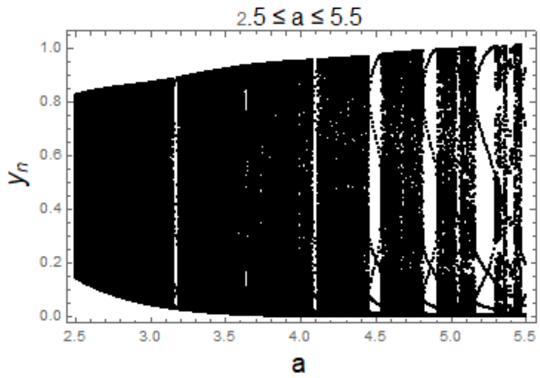

Fig.5.1(d) Bifurcations on y-axis for 2.5≤a≤5.5.

Here we have found the bifurcations over r.

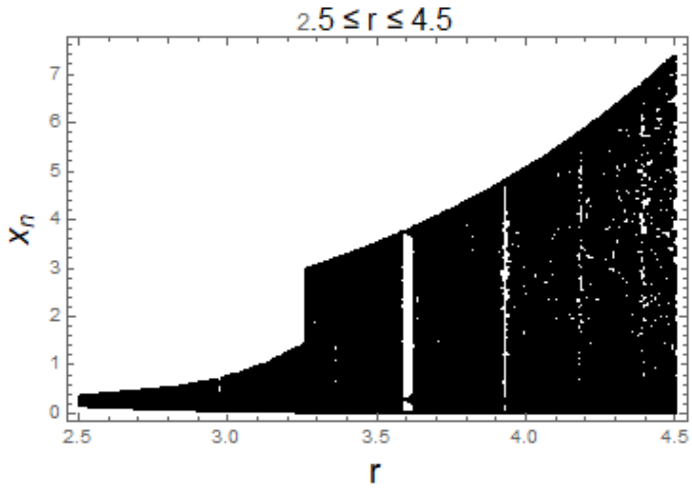

Fig.5.1(e) Bifurcations on x-axis for 2.5≤r≤4.5.

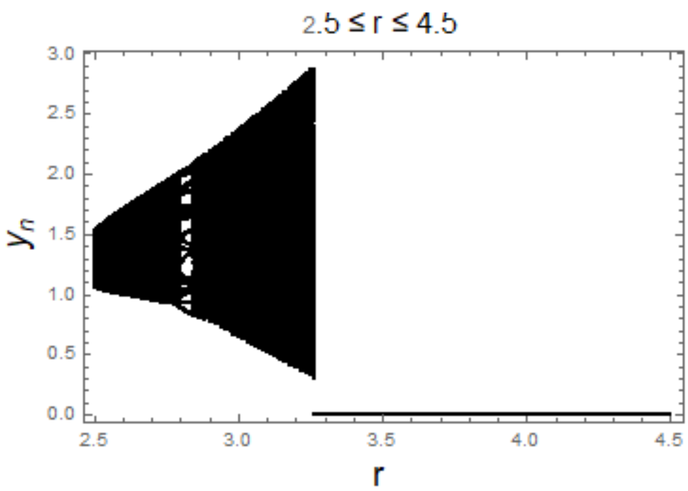

Fig.5.1(f) Bifurcations on y-axis for 2.5≤r≤4.5.

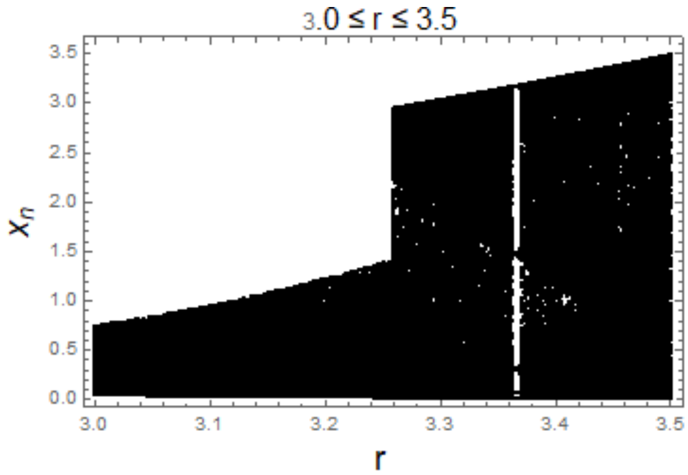

Fig.5.1(g) Bifurcations on x-axis for 3.0 ≤r ≤3.5.

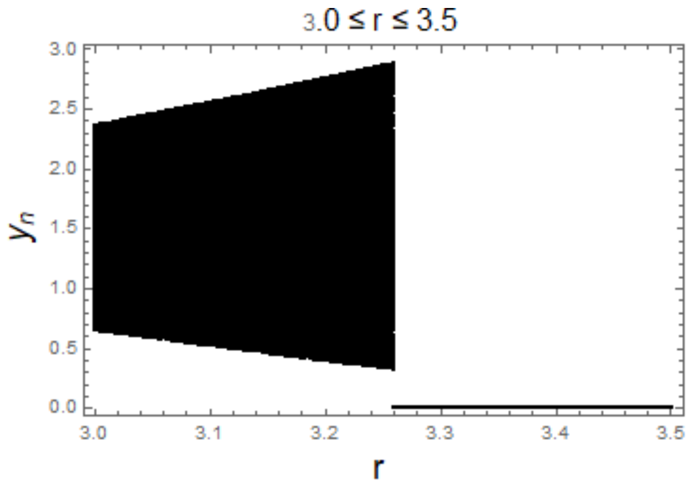

Fig.5.1(h) Bifurcations on y-axis for 3.0 ≤r ≤ 3.5.

We have performed calculations and obtained bifurcations displaying interesting features of evolution, shown in Fig.5.2. To obtain bifurcation diagrams of system (1), first we fixed a and varied r and then we fixed r and varied a. Following cases have been considered:

(a) a = 1 and 1.8 ≤ r ≤ 3.5 & 3 ≤ r ≤ 3.3, one finds period doubling type bifurcations leading to chaos. Also, it has been observed a 3–periodic windows following again a period doubling bifurcation has appeared in the figure in the close range of r (Fig.5.2(a)).

(b) a = 1.2 and 2.5 ≤ r ≤ 3.5, the bifurcations along x- & y- axes, are shown in Fig.5.2(b). These figures show very unusual and significantly different type of bifurcations than the earlier case.

(c) In the third case, we have varied a and fixed r; r = 2.5 and 1.3 ≤ a ≤ 2.5.

These diagrams clearly show the presence of chaos and complexity in the system. The middle row bifurcations are characteristic in the sense because one observes behavior: from period one to chaos, then period 5, then chaos and so on. The lowest row figures, again, display complicated form of bifurcation.

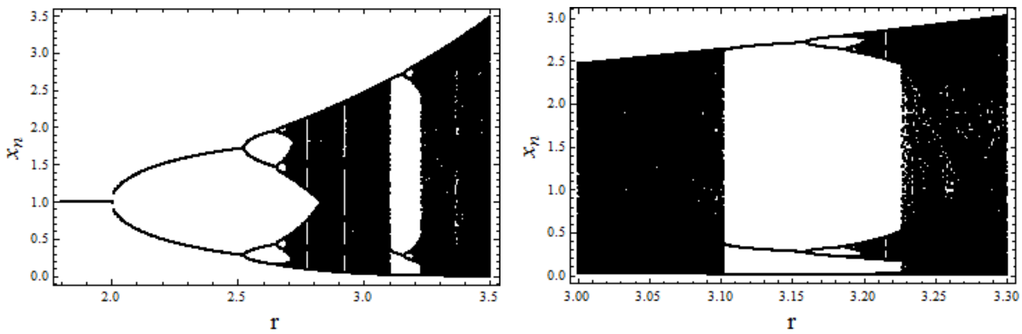

Fig.5.2(a) Bifurcations along x-axis for a = 1 and 1.8 ≤ r ≤ 3.5 & 3 ≤ r ≤ 3.3.

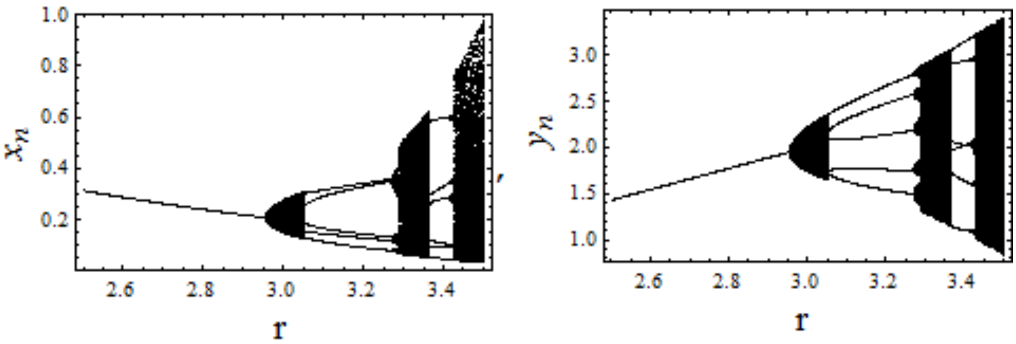

Fig.5.2(b) Bifurcations along x and y axis for a = 1.2 and 2.5 ≤ r ≤ 3.5.

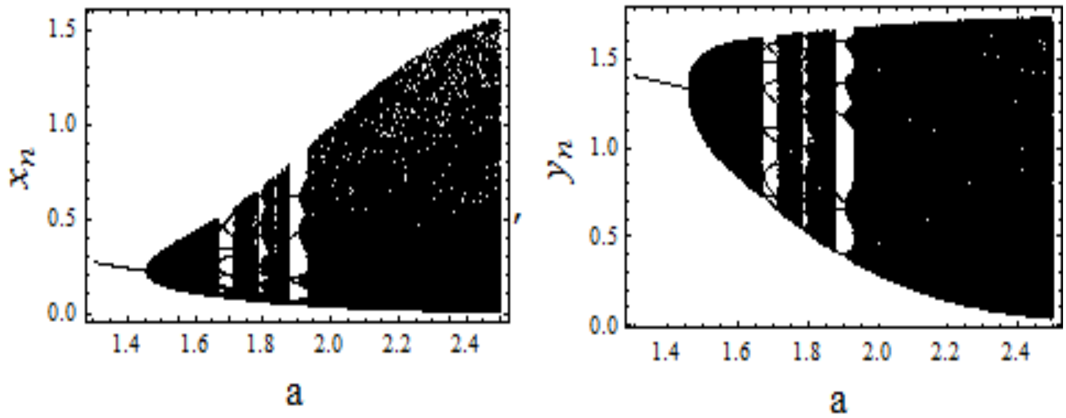

Fig.5.2(c) Bifurcations along x and y axis for r = 2.5 and 1.3 ≤ a ≤ 2.5.

5.3.2 Attractors

The regular as well as chaotic attractors have been obtained by varying value of parameter a. Qualitative change in structure of attractors, from regular to chaotic, have been observed during these processes. We have fixed ‘r’ as r = 2.8 and varied ‘a’ from a = 1.28 to 1.64. Initially, we have obtained a point attractor which, as ‘a’ increases, changes into closed curves, limit cycle attractors through Hopf bifurcation Fig. 5.3(a). Further increase in a, results into dense chaotic attractors, Fig. 5.3(b). Within the dense chaotic one finds the phenomena of bistability, folding and other complex structures, Fig. 5.4.

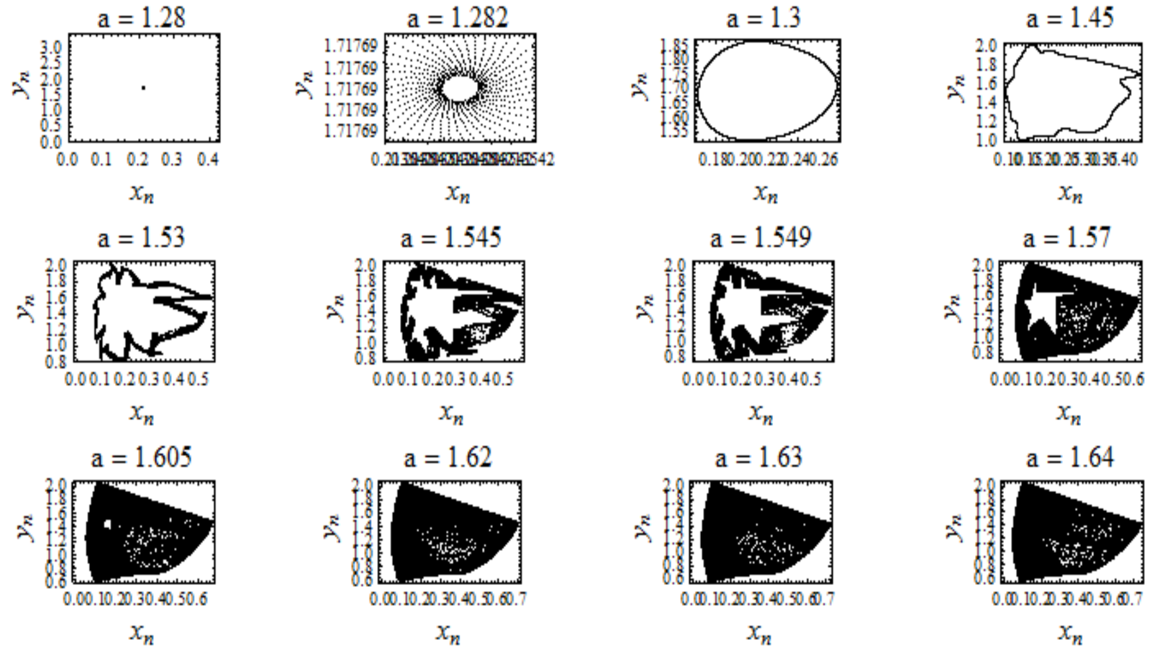

Fig 5.3 (a) Regular and chaotic attractors of map (1) for r = 2.8 and different values of a.

Attractors are formed through Hopf bifurcation and with initial condition $x_0 = 0.2$, $y_0 = 0.78$.

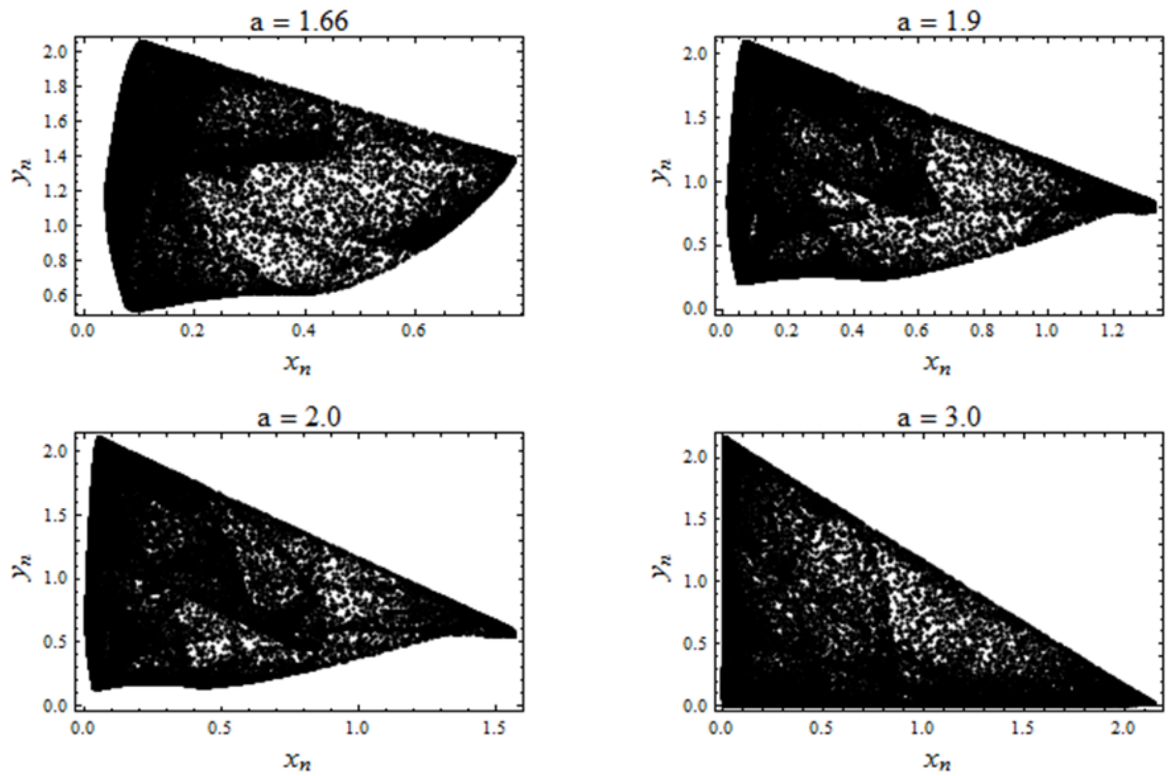

Fig. 5.3(b): Qualitative change in structure of chaotic attractors when r = 2.8 and a increases from a =1.66 to 3.0.

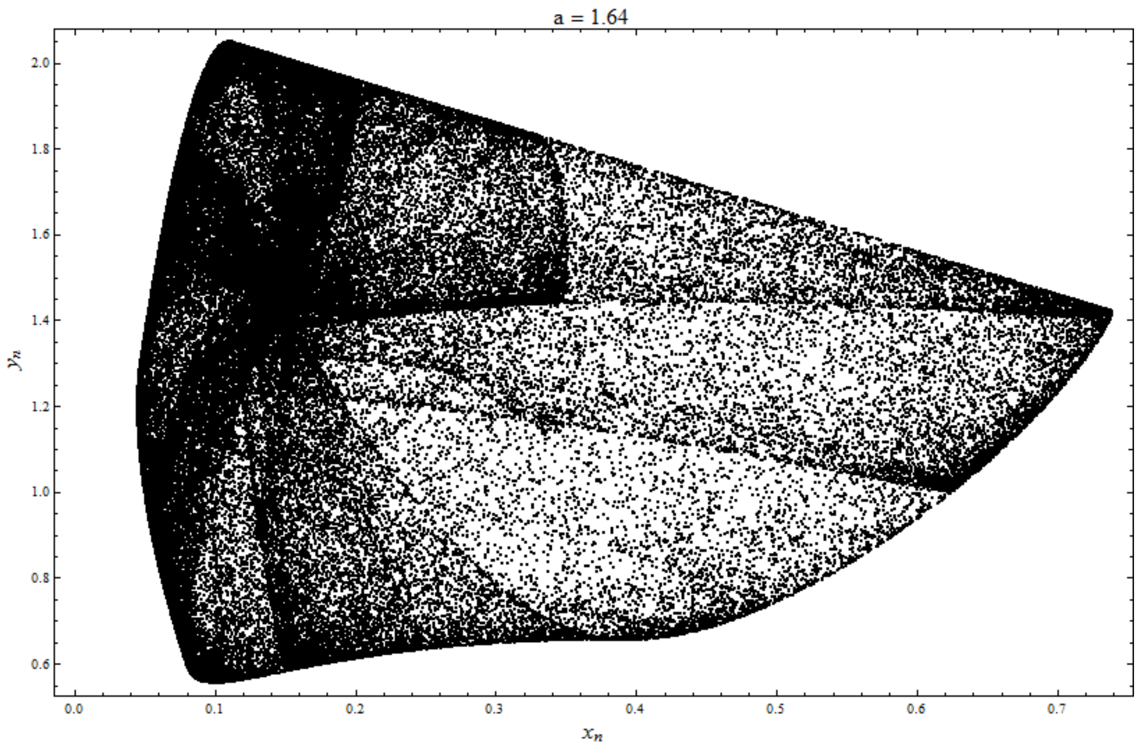

Fig. 5.4: Chaotic attractor of system (1) for r = 2.8 and a =1.64.

5.3.3 Lyapunov Exponents (LCEs)

Lyapunov exponents provide measure of chaos. Lyapunov exponents or Lyapunov characteristic exponents, named after the Russian engineer Alexander M. Lyapunov (Lyapunov, 1992), are quantities that characterize the rate of separation of trajectories that initiated infinitesimally close to each other (Lynch, 2007, Leonov and Kuznetsov, 2007, Grassberger and Procaccia, 1983b). Quantitatively, divergence of two trajectories in phase space with initial separation δx_0 and separation after n[th] iterations δx_n can be related, (provided that the divergence can be treated within the linearized approximation), by

$$|\delta x_n| \approx e^{\lambda n} |\delta x_0| \quad \text{or by} \quad |\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \qquad (2)$$

where $\lambda > 0$ is the Lyapunov exponent. The right side of (2) stands for continuous system. The system described by a map f be regular as long as $\lambda \leq 0$ and chaotic when $\lambda > 0$. This condition, known as sensitive dependence on initial conditions, is one of the few universally agreed-upon conditions defining chaos. Fixing r = 2.8, we have calculated maximum of LCEs at each iteration for our model and plotted them in Fig.5. 5. The figure 5.5(a) shows all LCEs are negative and so the evolution is regular. But, in the Fig.5.5(b) as all LCEs positive and so in this case the evolution is chaotic.

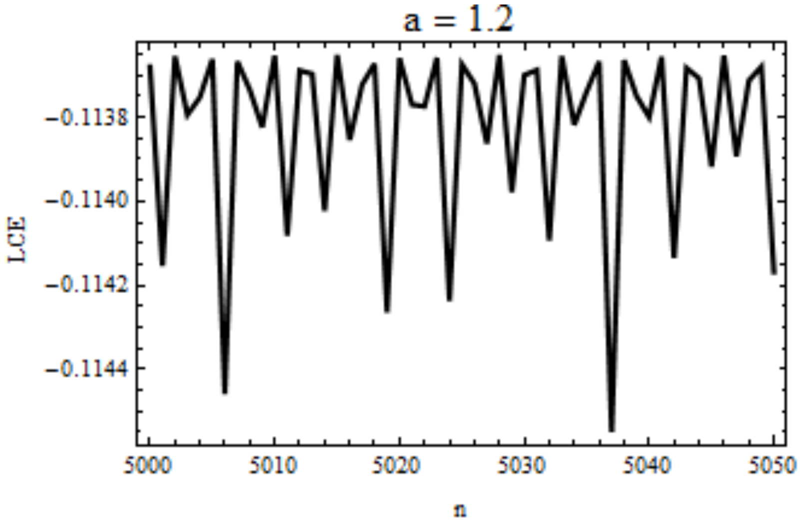

Fig.5.5(a) Regular motion with negative LCE's.

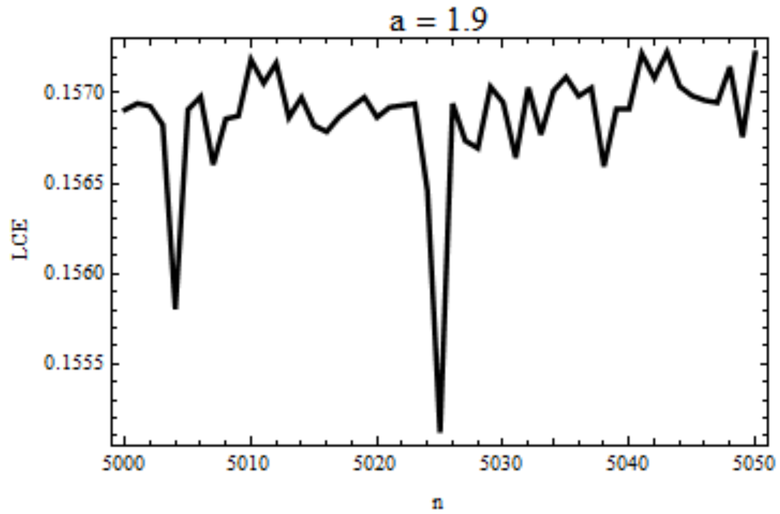

Fig.5.5: (b) Chaotic motion with positive LCE's.

5.3.4 Topological Entropies

The topological entropy discussed here is actually Kolmogorov-Sini entropy (Nagashima and Baba, 1998). As stated in the introduction, topological entropy provides certain measure of complexity and so, it is a simple indicator of complexity. For this system, numerical simulations have been performed to obtain the topological entropies. As shown in Fig. 5.6(a), for fixed value of parameter $r = 2.8$ and $1.0 \le a \le 4.5$, the system has enough positive entropy for $1.0 \le a \le 3.3$ and even more topological entropy in $3.3 < a \le 4.5$. Also, a 3-D plot of topological entropy is obtained and shown in Fig. 5.6(b) by varying both a and r; $1.5 \le a \le 4.0$ & $1.0 \le r \le 3.0$.

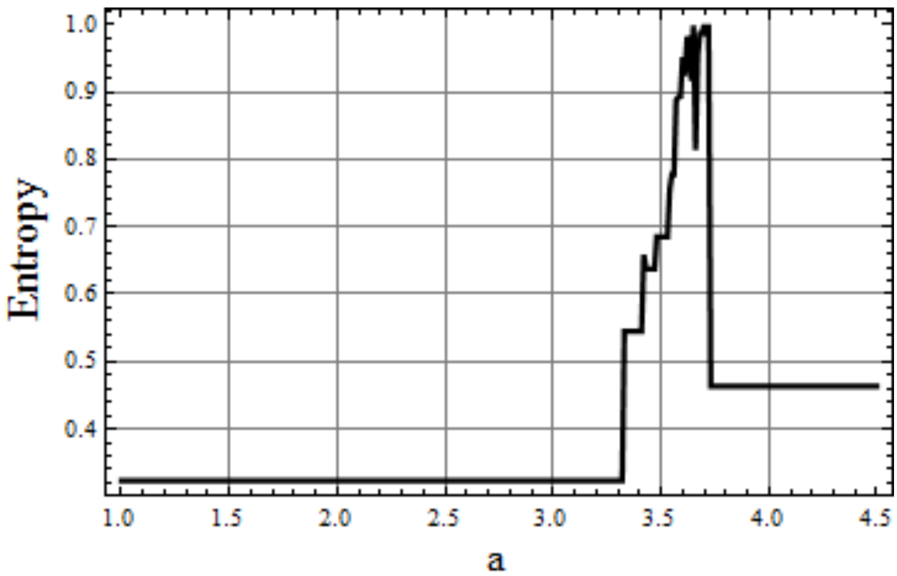

Fig. 5.6(a): Plot of topological entropy by fixing r = 2.8 and varying a, 1.0 ≤ a ≤4.5.

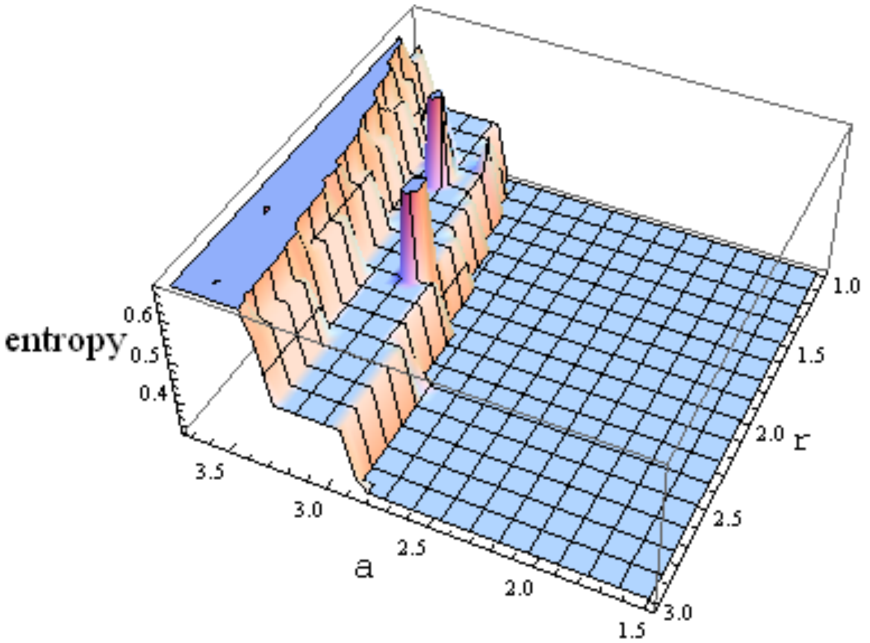

Fig. 5.6(b): 3-D plot of topological entropy for 1.5 ≤ a ≤ 4.0 & 1.0 ≤ r ≤ 3.0.

5.3.5 Correlation Dimension

Correlation dimension provides the dimensionality of the evolving system (Martelli, 2011, Grassberger and Procaccia, 1983a, Grassberger and Procaccia, 1983b). It is a kind of fractal dimension and its numerical value is always non-integer. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. The procedure to calculate correlation dimension is statistical and that is what we have followed here (Saha and Das, 2013). We have explained it in detail in chapter 2.

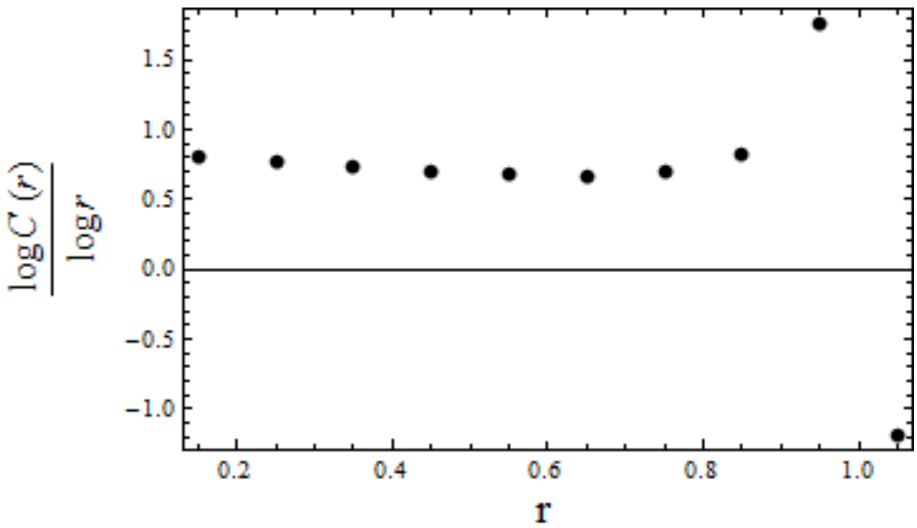

Fig.5.7: Plot of correlation curve for the chaotic plant-herbivore model with r = 2.8 and a = 1.64.

When we use the least square linear fit to the data of correlation integral, we obtain the equation of the straight line fitting the data as

$$Y = 1.03089 - 0.642896\, x$$

The y-intercept of straight line is 1.03089

So, the correlation dimension is obtained approximately as $D_c \approx 1.03$.

5.4 Discussion

Here we have worked upon a general plant-herbivore model, partly motivated by the dynamics of a gypsy moth infestation through biomass transfer from plants to the gypsy moth. The discrete 2-D model is controlled by the two parameters 'r', the nutrient uptake rate of the plant while 'a' is the amount of leaves eaten by the herbivore

We have studied the nonlinear behavior together with certain measure for chaotic evolution. Bifurcation diagrams of this model have been drawn by varying both the parameters a and r. These figures provide information regarding evolution with stable solutions as well as chaotic nature of nonlinear properties and limitation for parameter space. Regular and chaotic attractors have been drawn. Mathematical calculations and analyses of this model provide us with biological insights that may be used to devise control strategies to regulate the population of the herbivore.

Since the herbivore movements are random, it is more appropriate to study a stochastic model instead of deterministic one. Such realistic plant-herbivore system will be the objective of our research in near future.

Chapter 6

Dynamics of evolution of Mature Population

6.1 Introduction

Usually biological systems are complex and multicomponent. They are spatially structured and their individual elements possess individual properties. Such complexity also effects the system significantly during evolution. Natural processes tend to vary over time and space, as well as between the species. In recent years there has been a great emphasis on three concerning phrases: nonlinear dynamics, chaos and complexity. This interest has led to a large number of popular-science articles covering models and graphics to explain chaos, regularity and chaos control in certain cases. Henri Poincaré(1854–1912), a late-nineteenth century French mathematician was the first one to extensively study topology and topological systems. All-natural systems exhibit massive diversity. Complex systems are characterized by an internal structure which is built by numerous and varied processes, subsystems and interconnections. Systems featured by complexity display a number of properties such as uncertainty, interactions at different levels, self-organization and nonlinear feedback. Due to its nonlinear structure such systems may display the properties like complexity and chaos. Elaborate descriptions on complexity can be viewed from some well written articles (Weaver, 1948, Simon, 1962, Manson, 2001).

A chaotic system can be better understood by measuring components like Lyapunov exponents (LCEs), topological entropies, correlation dimension etc. Topological entropy, a non-negative number, provides a perfect way to measure complexity of a dynamical system. The concept of topological entropy was first introduced by Adler, Konhelm and McAndrew in the 1960s. For a system, more topological entropy signifies more complexity. Topological entropy measures the evolution of distinguishable orbits over time, thereby providing an idea of how complex the orbit structure of a system is (Adler et al., 1965, Bowen, 1973).Topological entropy describes the rate of mixing of a dynamical system. It is related to Lyapunov exponents both through the dependence of rate and through the ergodicity. For a system having non-zero topological entropy, the rate of mixing must be exponential which is comparable to Lyapunov exponents. Though such exponentiality is not relative to time, rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. The book by Nagashima and Baba (Nagashima and Baba, 1998) gives a very clear definition of entropy.

Another very important measure of chaos is LCEs. LCEs provide the rate of divergence of orbits which initially start very close to each other. Positive measure of Lyapunov exponents (LCEs) signifies presence of chaos (Benettin et al., 1980, Abarbanel et al., 1992).

While working with population dynamics in many models the increases in population due to birth are assumed to be time-independent, but that is not always the case. In many cases some species reproduce only during a single period of the year. We have worked with a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period. This model was proposed by (Tang and Chen, 2002). We have tried to obtain bifurcations, LCE's and entropy to analyze and measure chaos in the system. The sequence of bifurcations, leading to chaotic dynamics shows that the dynamical behaviors of the single species model with birth pulses are very complex and chaotic.

6.2 The Model

It is assumed that population size changes according to population growth ratio equation in absence of stage structure:

$$N = B(\acute{N})N - dN \qquad (1)$$

where d>0 is the death rate constant, and B(N)N is a birth rate function with B(N) satisfying the following basic assumptions for $N \in (0,\infty)$.

- $B(N) > 0$.
- B(N) is continuously differentiable with $B`(N) < 0$.
- $B(0+) > d > B(\infty)$.

Later it is assumed that the single species population in model (1) has stage structure, and that the population N is divided into immature and mature classes, with the size of each class given by x(t)and y(t), respectively, so that N(t)= x(t)+y(t), and only the mature population can reproduce. The model has been discretized leading to the following equations.

$$x_{n+1} = x_n e^{-(\delta+d)} + b[y_n + x_n(1-e^{-\delta})]e^{-[d+e^{-d}(x_n+y_n)]}$$
$$y_{n+1} = e^{-d}(1-e^{-\delta})x_n + e^{-d}y_n \qquad (2)$$

6.3 Numerical Simulations

Performing various numerical simulations, the dynamics of evolution have been investigated by obtaining bifurcation diagrams, calculating Lyapunov exponents, topological entropy and correlation dimensions of the system for different cases.

6.3.1 Bifurcations

As we have already discussed in the previous chapters that bifurcations play a very important role in studying the dynamics of any system. We have found bifurcations of this system which are shown below.

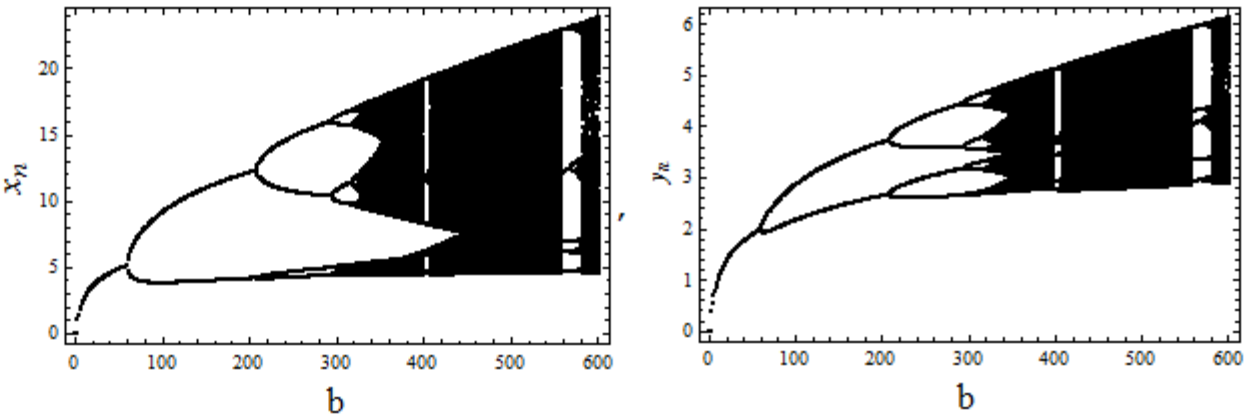

Fig.6.1: Bifurcation diagrams along x- and y- axes of system (2) for 0 ≤ b ≤ 600. when d= 0.7 and δ = 0.5 in fig.6.1 showing period doubling route to chaos.

6.3.2 Periodic attractors

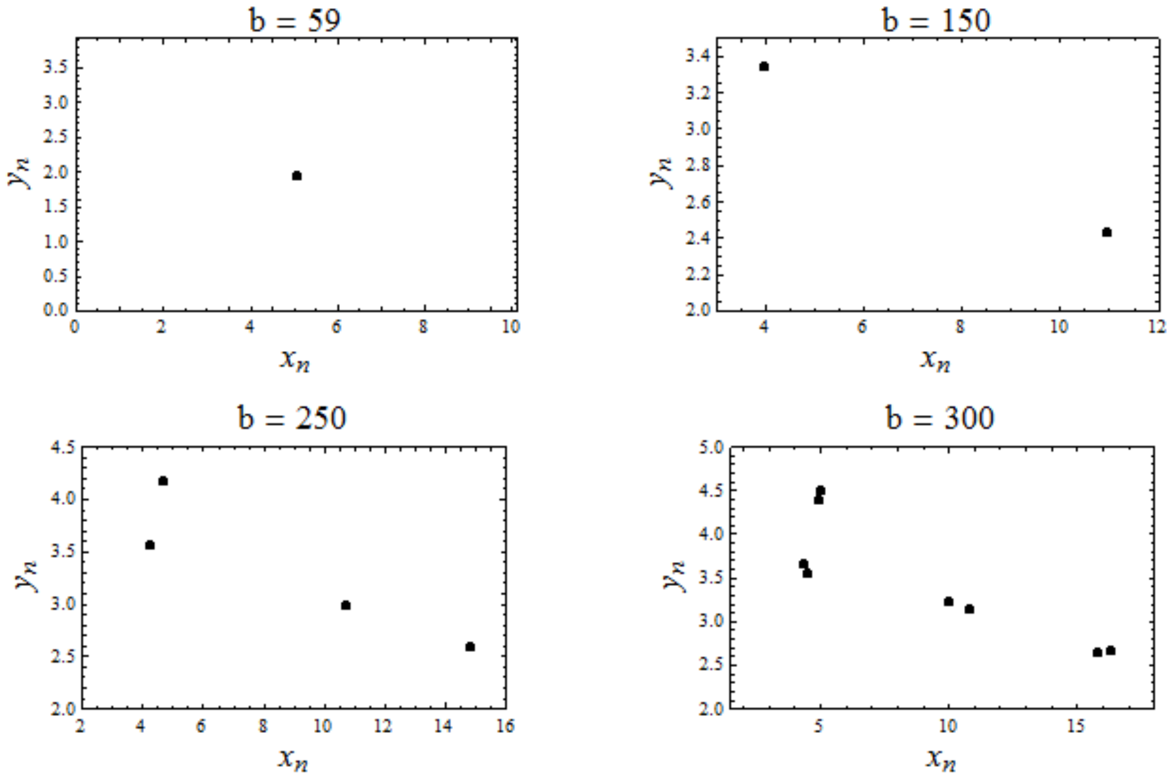

Fig. 6.2: Periodic attractors of system (2) with periods 1, 2, 4, 8 for different values of parameter b. Other parameters are d= 0.7 and δ = 0.5.

6.3.3 Chaotic Attractors

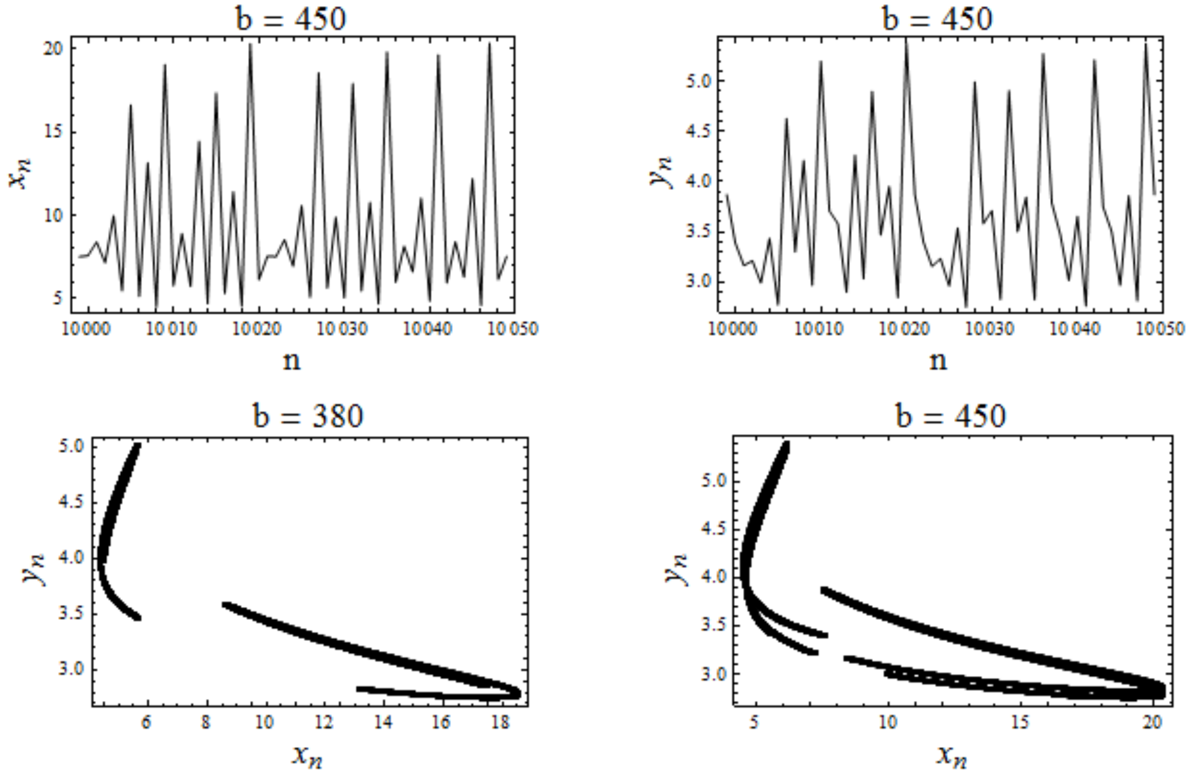

Fig.6.3: Chaotic time series plots, (upper row), and plots of chaotic attractors, (lower row) are shown here. The other parameters are taken as d= 0.7 and δ = 0.5.

6.3.4 LCEs and Topological Entropy

Lyapunov exponents can be considered as generalizations of the eigenvalues of steady-state and limit-cycle solutions to differential equations. The eigenvalues of a limit cycle characterize the rate at which nearby trajectories converge or diverge from the cycle. The Lyapunov exponents do the same thing, but for arbitrary trajectories, not just the special ones that are periodic. Calculation of Lyapunov exponents involves (for nonlinear systems) numerical integration of the underlying differential equations of motion, together with their associated equations of variation. The topological entropy measures the growth of the number of periodic points. Similar to Lyapunov exponent, it also measures how "complex" the map is.

In communication theory, a measure of uncertainty or randomness is a factor that is related to information. The greater the entropy, the greater the uncertainty, and the greater the amount of information that could be transmitted. However, once received, information represents a decrease in uncertainty. The flip of a fair coin yields one bit of entropy with 0 or 1 representing heads or tails. A binary bit with equiprobability of a 0 or 1 is random. Also, the higher the probability of an event or a state, the lower the entropy. Back in 1948, Claude Shannon helped establishing the theoretical basis for the development of information and communication theory with his equation of entropy.

A second intuitive interpretation of entropy is as a measure of the disorder in a system. There are interesting examples of systems that appear to develop more order as their entropy (and temperature) rises. These are systems where adding order of one, visible type (say, crystalline or orientationally order) allows increased disorder of another type (say, vibrational disorder). Entropy is a precise measure of disorder but is not the only possible or useful measure. Topological entropy is a nonnegative number which measures the complexity of the system. Roughly, it measures the exponential growth rate of the number of distinguishable orbits as the time advances.

To name a periodic orbit, we need to choose one of its cyclic permutations only. The number of distinct periodic orbits grow rapidly with the length of the period. A simple indicator of the complexity of a dynamical system is its topological entropy. In the one-dimensional setting, the topological entropy is a measure of the growth of the number of periodic cycles as a function of the symbol string length (period).

The LCEs for the given system are shown below:

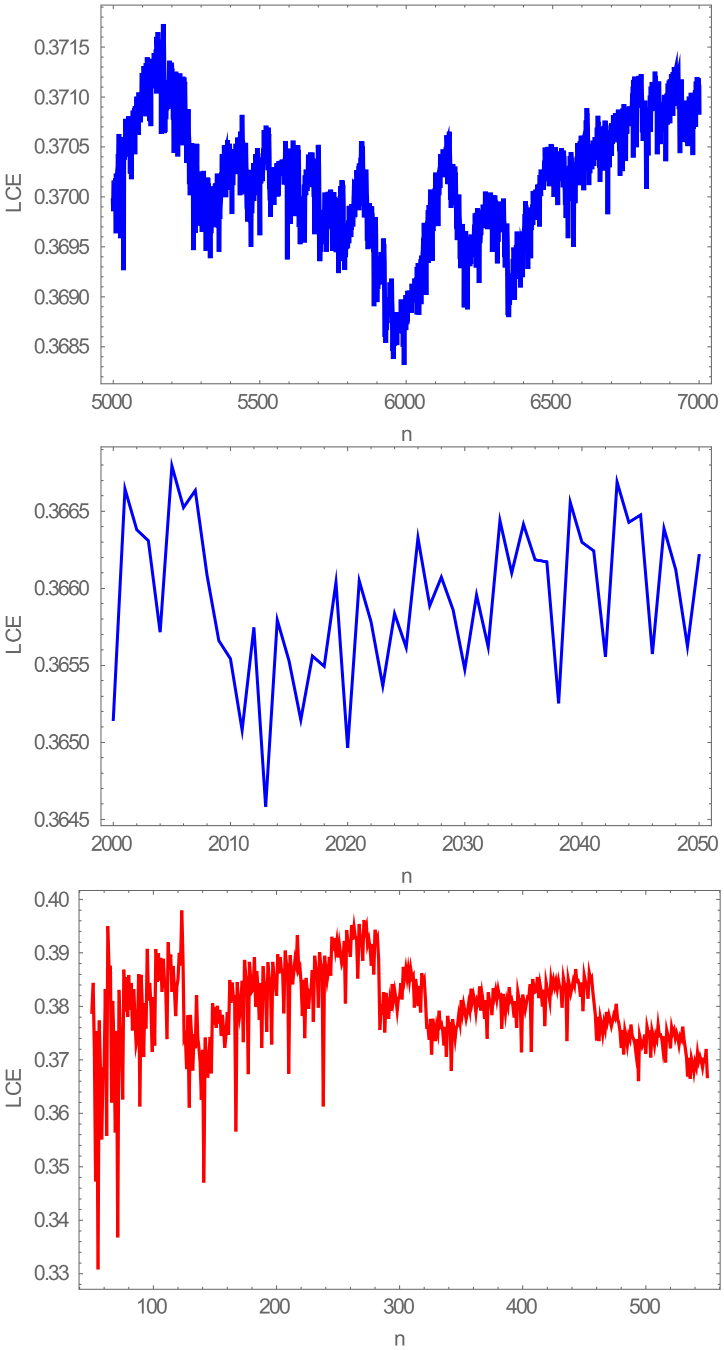

Fig.6.4(a) LCE's for δ=0.5, b=540 & d=0.7.

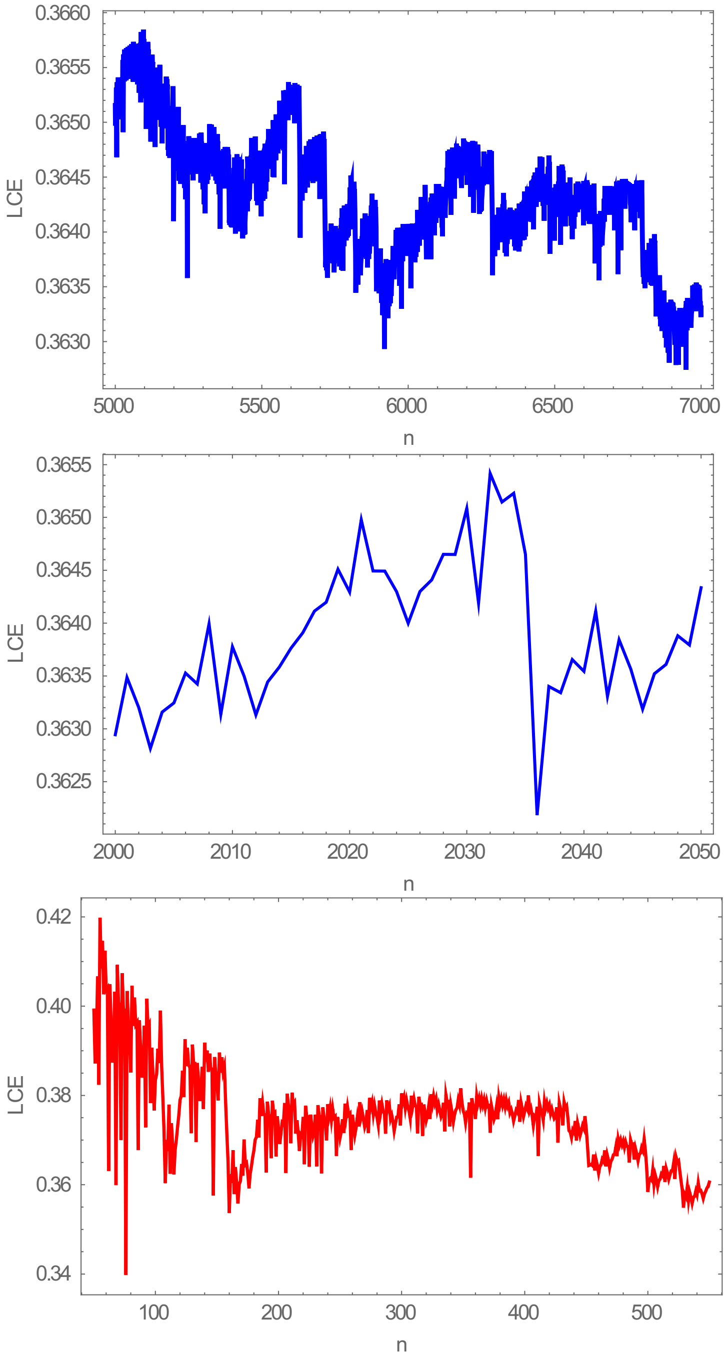

Fig.6.4(b): LCE's for δ=0.5, b=480 & d=0,7.

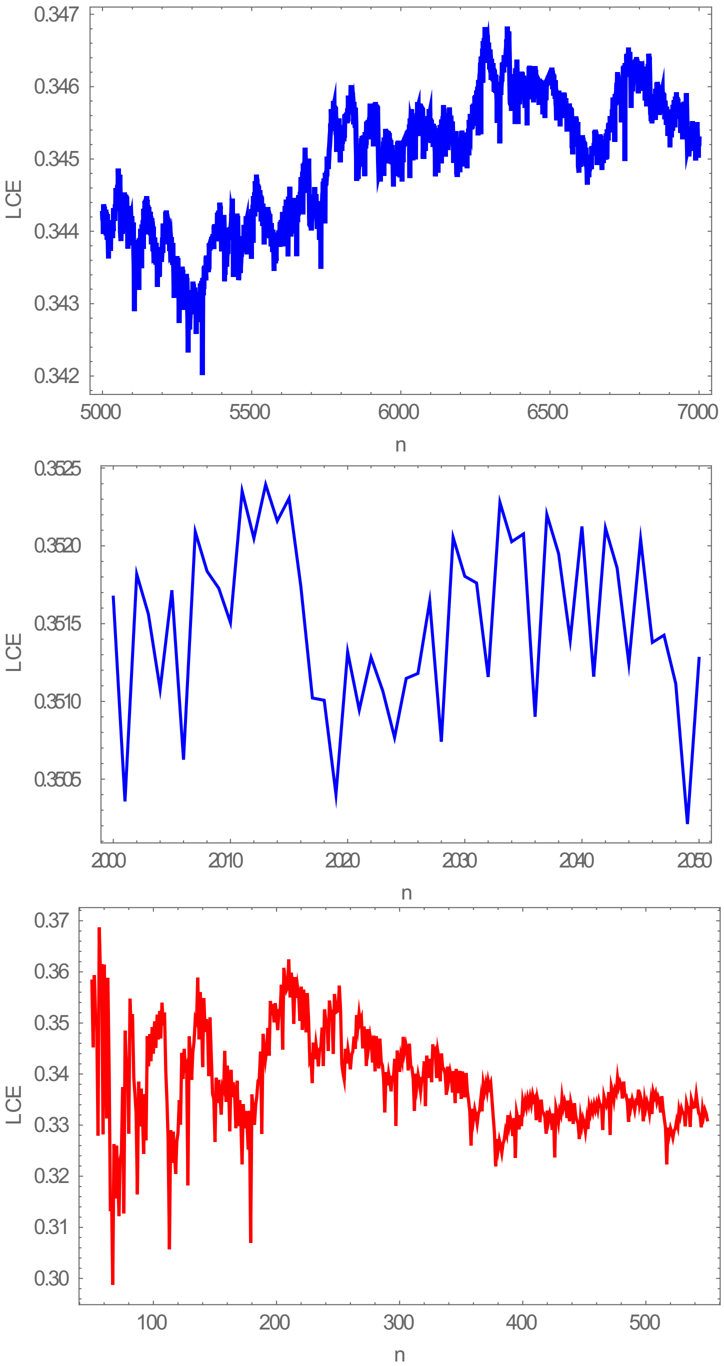

Fig.6.4(c): LCE's for δ=0.5, b=600 & d=0,7.

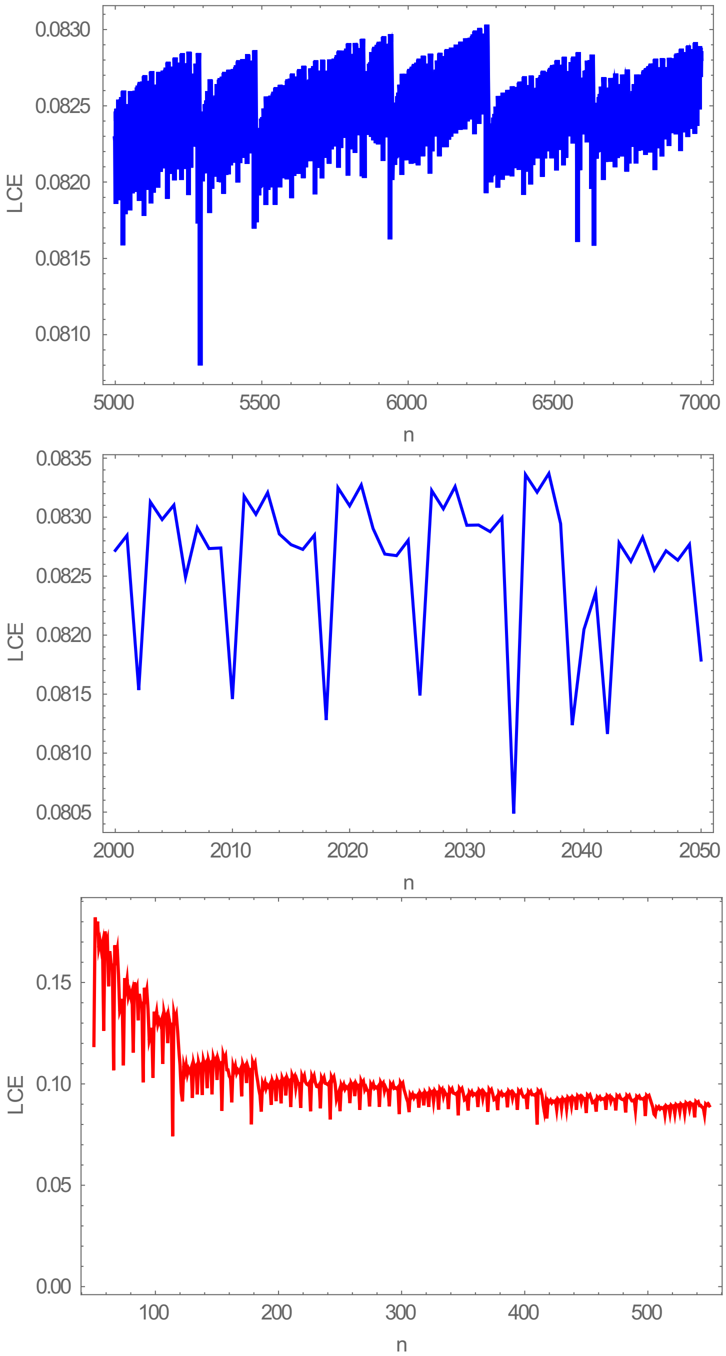

Fig.6.4(d): LCE's for δ=0.5, b=330 & d=0,7.

Measure theoretic entropy, which is also called the Kolmogorov—Sinai invariant, was defined for measure preserving transformations of probability measure spaces. The concept of topological entropy was first introduced by Adler, Konhelm and McAndrew in the 1960s.

For a system having non-zero topological entropy, the rate of mixing must be exponential which is comparable to Lyapunov exponents. Though such exponentiality is not relative to time, rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. The graphics for Topological Entropy are shown below:

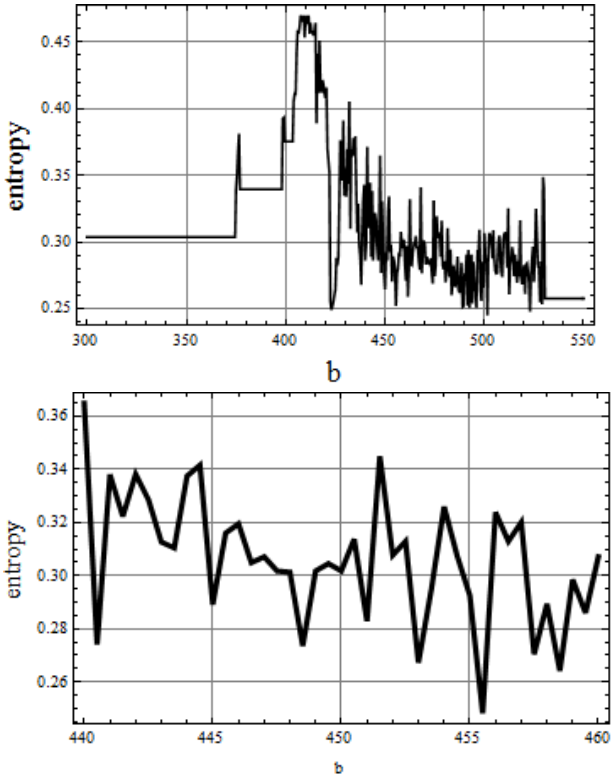

Fig.6.5: Plots of topological entropies are presented with d= 0.7 and δ = 0.5 and 300 ≤ b≤ 550 and 440 ≤ b≤ 460.

6.3.5 Correlation dimension

Correlation dimension provides the dimensionality of the evolving chaotic attractor. It is a kind of fractal dimension and its numerical value is always non-integer. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity of the underlying attractor of the system. The procedure to calculate correlation dimension that we have followed here is statistical (Martelli,1999).

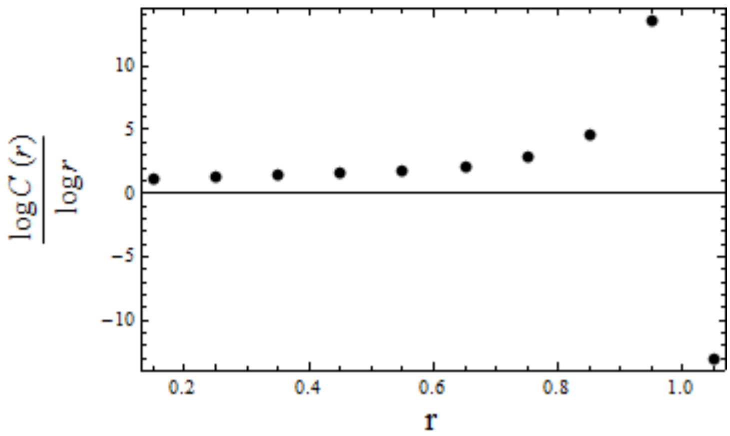

Fig.6.6: Plot of correlation integral curve for d= 0.7, b = 540 and δ = 0.5

When we use the least square linear fit to the data of correlation integral, we obtain the equation of the straight line fitting the data as

$$Y = 2.4774 - 01.23719\ x$$

The y-intercept of straight line is 2.4774

Thus, the correlation dimension is obtained approximately as $D_c \approx 2.45$.

6.4 Discussion

Complexity and chaotic evolutionary motion have been discussed for discrete mature population model. Bifurcation diagrams, Fig.6.1, show the system evolves through a period doubling root to chaos. Measures of complexity; such as Lyapunov exponents, topological entropies, correlation dimension have been calculated and shown through figures, Fig.6.3 – Fig.6.6. Plots of LCEs and topological entropies show clearly the complexity nature of the system. The correlation dimension for the chaotic attractor when parameters are d= 0.7, b = 540 and δ = 0.5, is obtained approximately as, $D_c \approx 2.45$.

Chapter 7

Summary and scope in future

7.1 Chapter wise summary

Chapter 1 includes the basic concepts and definitions related to the work done in the present. We have tried to build a strong base and foundation for studying non-linear dynamics by explaining all the needed definitions and fundamental concepts.

Chapter 2 illustrates the methodology used which includes Bifurcations, LCEs, Topological entropy and correlation dimension. These all are the measures of chaos in a system. A deep understanding of these measures is a must to understand the concept of chaos or to measure chaos in a system or in comparing different chaotic systems. We have also described some newly discovered chaos indicators like FLI, SALI and DLI. For all calculations, we have used the MATHEMATICA software.

Chapter 3 includes the complexity measure in a simple type food chain system that has been investigated both analytically and numerically. In Ecological systems, food webs or food chains are constituted of several layers such that the consumers which eat from the bottom resource layer are the prey of another predator. This is due to interdependence of species within the system. So, the evolutionary dynamics of food chain webs are of complex nature and their dynamics are highly interesting. Small variations of parameters of the system show very divergent results. Models describing food chain can be obtained in the form of a set of ordinary differential equations or as a set of discrete form of equations. These models are obtained by observing the group behaviour of involving species, e.g. functional group behaviour.

We have worked with a model introduced by Deng. Regular and chaotic motions have been observed for certain sets of values of a parameter of the system. For more detailed study, the continuous model of food chain has been transformed into discrete model by using Euler's method.

Various measurable quantities for emergence of chaos, like Lyapunov exponents, topological entropies, correlation dimensions, have been numerically calculated and represented through plots. Finally, the chaos indicator, named Dynamic Lyapunov Indicator (DLI), has been used to identify clearly the chaotic and regular motion.

In Chapter 4 we have talked about the evolutionary dynamics of a two-gene model for chemical reactions corresponding to gene expression and regulation. The studies performed here deal with a two-gene Andrecut-Kauffman model. In this two-dimensional discrete system, dynamical variables describe the evolution of the concentration levels of transcription factor proteins.
We intended to investigate certain dynamic behaviors of the system for evolution showing irregularities due to presence of chaos and complexity. To study the characteristics of complex nature of evolutionary phenomena, bifurcation diagrams have been drawn by varying a certain parameter. Then numerical investigations are performed to obtain Lyapunov exponents (LCEs), topological entropies and correlation dimensions for different sets of parameters of the system. Results obtained are shown through graphics. Finally, the complex nature of evolutions has been discussed on the basis of results obtained through this study.

Chapter 5 talks about a simple host-parasite type model that has been considered to study the interaction of plants and herbivores. Plant-herbivore interactions have a very important role in our environment. Since plants and associated herbivores constitute more than half of the eukaryotic species inhabiting our world, the understanding of this relationship between animals and plants is extremely important for land management. The removal of a particular plant species or group may result in the disappearance of many animals from an area. This argument provides a strong basis for studying animal plant relationships and infer results for future.

Plant-herbivore model is a generalization of the host-parasite and Nicholson-Bailey models, studied under various assumptions and modification. More extensive studies have been carried out recently on Plant-herbivore model by formulating a suitable mathematical model and some interesting outcomes on the evolutionary behavior have been pointed there.

Perceptive bifurcation diagrams, which give insightful results, have been presented here showing chaos and complexity in the system during evolution. Measure of complexity and chaos in the system is explained by performing numerical calculations and obtaining Lyapunov exponents, topological entropies and correlation dimension. Results are displayed through interesting graphics.

Chapter 6 talks about a stage-structured predator-prey model with impulses, that is investigated. It is a known fact that the discrete nonlinear age and stage-structured population models serve as great tools for studying the dynamics of various ecological populations. Such models contain species which may possess diverse life histories. In this work, we worked upon a single-species model with stage structure for the dynamics in a wild animal population for which births occur in a single pulse once per time period.

7.2 Future Outlook

In future, I would like to extend my studies to problems relating to evolution of other biological species and to problems on various types of epidemics. With similar as well as some new procedures applicable to stochastic and deterministic models, one can derive many results which are interesting and potentially useful. Statistical measures like probability, significance of variance etc. can be used in the extended study. My planning is also to develop better techniques and codes of software, (e.g. MATHEMATICA), which provide opportunity for more detailed analysis. Applications of chaos theory are widespread across biology, chemistry, physics, economics, mathematics and among many other fields. Often, systems with a large number of coupled variables exhibit chaotic behaviour, including weather systems, job markets, population dynamics and celestial mechanics. So, it will be a good idea to work with some more models with an attempt to do some inter-disciplinary research where we can merge fields like Physics, Economics, Biology etc. to formulate a model and further study it for chaos.

My future aim of further study will intend in these areas. We might also work in the direction of generalized modelling where we can use our generalized results to compare a few systems, their stabilities and chaoticity. It is a universal approach for investigating dynamics in nonlinear systems. In these models the processes under observation are not restricted to specific functional forms. That is the reason that a single generalized model can describe systems which share a similar structure. Even though the concept of generality is there, still it helps us to study the dynamical properties of models more efficiently in the framework of local bifurcation theory.

The success, which will be achieved in future study, depends on how accurately and appropriately the Mathematical models will be formulated and how accurately the numerical simulations will be performed.

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